THEF-NIJM 09.05

# Spinor Solutions and Massless limit for Spin 3/2

J.W. Wagenaar and Th.A. Rijken

*Institute for Mathematics, Astrophysics, and Particle Physics,*
*University of Nijmegen, Nijmegen, the Netherlands*

## Abstract

In these notes we use the construction of a field theory for the spin 3/2 fields, using an extended auxiliary-field formalism of the kind III. The present version is henceforth refered to as Model-V. In order to impose sufficient constraints on the $\psi_\mu$-fields one spinor auxiliary field $\chi(x)$ was needed. Here, we solve the field equation for the one-particle states, i.e. we solve the inhomogeneous Dirac equation for $\psi^\mu(x)$ and study the massless limit for its solutions. Using these solutions, we derive again the one-particle propagator, both for the massive and the massless case. It is shown explicitly that the "wrong" helicities decouple for the massless case, when coupled to a conserved current.

## I. INTRODUCTION

In our work on the quantization of the spin-3/2 fields [1, 2] we described a formalism for the spin-3/2 fields, employing the device of an auxiliary field-formalism to satisfy the constraints. The Lagrangian reads

$$\mathcal{L}_\chi \;=\; \mathcal{L}_{3/2} + M_{3/2}\bar{\chi}\gamma^\mu\psi_\mu + M_{3/2}\bar{\psi}_\mu\gamma^\mu\chi + bM_{3/2}\bar{\chi}\chi \ . \tag{1.1}$$

with

$$\mathcal{L}_{3/2} \;=\; -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\rho(\partial_\sigma\psi_\nu) + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\partial_\sigma\bar{\psi}_\mu)\gamma_5\gamma_\rho\psi_\nu - M_{3/2}\bar{\psi}_\mu\sigma^{\mu\nu}\psi_\nu \ . \tag{1.2}$$

It turned out that a covariant spin-3/2 propagator in this formalism requires that the parameter $b = 0$. In this paper we restrict our discussion to that case. Furthermore, we denote in the following the mass by $M_{3/2} \equiv M$.

The main purpose of this paper is to demonstrate the massless limit of the spin-3/2 propagator by using the explicit solutions of the Dirac equation. This, in order to see the conditions under which the "wrong" helicities decouple in the massless limit. It is well known [3] that the so-called 'little group' $L(\overset{o}{p})$ the four-vector $p^\mu$ is different for the massive case ($L(\overset{o}{p}) \cong SO(3)$) and for the massless case ($L(\overset{o}{p}) \cong E(2)$) and therefore have different ireducible representations. In the massive- and massless- case the dimension is $2S + 1$ and 2 respectively, where S denotes the spin of the particles. In appendix A we describe the Lie algebra of these "little groups".

The contents of this paper is as follows. In section II we give the coupled Dirac equations for the spin-3/2 spinors and the spin-1/2 auxiliary spinors. In section III we give the explicit form of the solutions of the spinors on momentum-space. In section IV the massless limits are described using the cartesian base, and in section V the same but now in the sperical-base. In section VI the propagator in the massless limit is studied and the projection operators $\Lambda^{\mu\nu}(p)$ are described using the explicit forms of the Dirac spinors of the previous sections. These are compared with those derived in [2] and shown to agree except for a so-called "dipole-ghost" term. In section VII we show that the "dipole-ghost" term can be eliminated from the propagator exploiting the "gauge-symmetry" which applies in the massless case. Finally, we finish this paper by some conclusions in section VIII. In appendix A we describe the group theoretical difference between the massive and massless case, showing that the Lie algebra of the so-called "little group" of the four-vector $p^\mu$ gives the $SO(3)$ algebra in the massive case, and in the massless case the Lie algebra of the Euclidean group in two dimensions $E_2$. The latter has the consequences that for the massless case only the helicities $\lambda = \pm J$, where J is the spin of the particle.

## II. ONE-PARTICLE SOLUTION

The one-particle wave-functions, corresponding to the $\psi^\mu$- and $\chi$-fields satisfy the following Dirac equations:

$$\begin{aligned} &\left(\not{p}_{op} - M\right)\ \psi^\mu(x) = -M\gamma^\mu\ \chi(x)\ , \\ &\left(\not{p}_{op} - 2M\right)\ \chi(x) = 0\ , \end{aligned} \tag{2.1}$$

and the constraints

$$(i)\ \gamma\cdot\psi(x) = 0 \quad , \quad (ii)\ p_{op}\cdot\psi(x) = -\not{p}_{op}\chi(x) = -2M\chi(x)\ . \tag{2.2}$$

In terms of the -space wave-functi ns we have for $\chi(x)$ the spectral representation

$$\chi(x) = \int \frac{d^4p}{(2\pi)^4}\delta(p^2-4M^2)\ u_\chi(p)\ e^{-ip\cdot x}\ . \tag{2.3}$$

with

$$(\not{p}-2M)\ u_\chi(p) = 0\ \ (p^2 = 4M^2)\ . \tag{2.4}$$

The solutions for $\psi^\mu$ can be written as

$$\begin{aligned} \psi^\mu(x) &= \psi^\mu_0(x) + \hat{\psi}^\mu(x)\ , \\ \left(\not{p}_{op} - M\right)\ &\psi^\mu_0(x) = 0\ . \end{aligned} \tag{2.5}$$

Here, $\hat{\psi}^\mu$ is the special solution of the inhomogeneous Dirac equation for $\psi^\mu$, see (2.1). The special solution is easily found to be

$$\hat{\psi}^\mu = -M\left(\not{p}_{op} - M\right)^{-1}\gamma^\mu\ \chi(x) \tag{2.6}$$

From the spectral representation (2.3) we get, writing

$$\hat{\psi}^\mu(x) = \hat{u}(p)e^{-ip\cdot x} \quad , \quad \chi(x) = u_\chi(p)e^{-ip\cdot x}\ \ (p^2 = 4M^2)\ , \tag{2.7}$$

the solution

$$\begin{aligned} \hat{u}^\mu(p;s) &= -M\frac{(\not{p}+M)}{p^2-M^2}\gamma^\mu u_\chi(p,s) \\ &= -\frac{M}{p^2-M^2}\left(-\gamma^\mu\not{p} + 2p^\mu + M\gamma^\mu\right)\ u_\chi(p,s) \\ &= -\frac{1}{3M}\left(-M\gamma^\mu + 2p^\mu\right)\ u_\chi(p,s)\ \ (p^2 = 4M^2)\ . \end{aligned} \tag{2.8}$$

One easily verifies that this special solution satisfies the constraints (2.2). *Notice the special notation $\hat{u}(p;s)$, which indicates that it is made from the spinor $u_\chi(p,s)$. The latter has $s_z = s$, whereas $s$ for $\hat{u}$ only serves as an index without such meaning. In passing we note also that $s$ must be independent of $\mu$ in order that (2.8) satisfies the constraints of (2.2).*

## III. SOLUTIONS HOMOGENEOUS DIRAC EQUATION

Since both $\psi^\mu$ and $\hat{\psi}^\mu$ satify the two constraints (2.2), it follows that $\psi^\mu_0$ satisfies the two homogeneous constraints

$$(i)\ \gamma\cdot\psi_0(x) = 0 \quad , \quad (ii)\ p_{op}\cdot\psi_0(x) = 0\ . \tag{3.1}$$

This means that we can identify $\psi^\mu_0(x) = \psi^\mu_{RS}(x)$, i.e. the free Rarita-Schwinger spin-3/2 spinor with mass M. In momentum space, we denote the Rarita-Schwinger spinor by

$U^\mu(p,s)$ in the following. Using the stadard spin-1 polarization vectors for $p^\mu = (E_p, 0, 0, p)$ with $p^2 = M^2$ as

$$\epsilon^\mu(\pm 1) \;=\; \frac{1}{\sqrt{2}}\,(0, \mp 1, -i, 0) \quad , \quad \epsilon^\mu(0) = \frac{1}{M}\,(p, 0, 0, E_p) \ , \tag{3.2}$$

one has

$$\begin{aligned}
U^\mu(p, +3/2) \;&=\; u(p, +1/2)\ \epsilon^\mu(+1) \ , \\
U^\mu(p, +1/2) \;&=\; \sqrt{\frac{1}{3}} u(p, -1/2)\ \epsilon^\mu(+1) + \sqrt{\frac{2}{3}} u(p, +1/2)\ \epsilon^\mu(0) \ , \\
U^\mu(p, -1/2) \;&=\; \sqrt{\frac{2}{3}} u(p, -1/2)\ \epsilon^\mu(0) + \sqrt{\frac{1}{3}} u(p, +1/2)\ \epsilon^\mu(-1) \ , \\
U^\mu(p, -3/2) \;&=\; u(p, -1/2)\ \epsilon^\mu(-1) \ .
\end{aligned} \tag{3.3}$$

For the spin-1/2 spinors we take ($p^2 = M^2$) [see e.g. Carruthers]

$$u(p,s) = \sqrt{2M} \begin{pmatrix} \cosh \frac{1}{2}\zeta \\ 2s \sinh \frac{1}{2}\zeta \end{pmatrix} \otimes \varphi_s \quad (s = \pm 1/2) \ . \tag{3.4}$$

We recall that

$$\begin{aligned}
\cosh \zeta = \frac{E}{M} \quad &, \quad \sinh \zeta = \frac{p}{M} \ , \\
\cosh \frac{1}{2}\zeta = \sqrt{\frac{E+M}{2M}} \quad &, \quad \sinh \frac{1}{2}\zeta = \sqrt{\frac{E-M}{2M}} \ ,
\end{aligned} \tag{3.5}$$

so that the spinors in (3.4) can be written as

$$u(p,s) = \begin{pmatrix} \sqrt{E+M} \\ 2s\sqrt{E-M} \end{pmatrix} \otimes \varphi_s \quad (s = \pm 1/2) \ , \tag{3.6}$$

which differ by a factor $\sqrt{2M}$ from those of Bjorken & Drell.

## IV. THE MASSLESS LIMITS

(i) For the spin-1/2 spinors we get from (3.6) we have in the massless limit

$$u(p,s)|_{M=0} = \sqrt{p} \begin{pmatrix} 1 \\ 2s \end{pmatrix} \otimes \varphi_s \quad (s = \pm 1/2) \ . \tag{4.1}$$

(ii) In the case of the Rarita-Schwinger spinors we find for very small M

$$\begin{aligned}
&U^\mu(p, +3/2) \to \sqrt{p} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \varphi_+ \cdot \epsilon^\mu(+1) \quad , \quad U^\mu(p, -3/2) \to \sqrt{p} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \varphi_- \cdot \epsilon^\mu(-1) \ , \\
&U^\mu(p, +1/2) \to \sqrt{\frac{1}{3}} \sqrt{p} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \varphi_- \cdot \epsilon^\mu(+1) + \frac{1}{M} \sqrt{\frac{2}{3}} \sqrt{p} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \varphi_+ \cdot \left[M \epsilon^\mu(p,0)\right]_{M=0}
\end{aligned} \tag{4.2}$$

We notice that

$$[M\epsilon^\mu(p,0)]_{M=0} = (p,0,0,p) = p^\mu(M=0) \ ,$$

which is useful in the following. For $s_z = +1/2$, per component we have

$$\begin{aligned}
U^0(p,+1/2) &\rightarrow \frac{1}{M}\sqrt{\frac{2}{3}}p^{3/2}\begin{pmatrix}1\\1\end{pmatrix}\otimes\varphi_+ \ , \\
U^1(p,+1/2) &\rightarrow -\frac{1}{\sqrt{6}}\sqrt{p}\begin{pmatrix}1\\-1\end{pmatrix}\otimes\varphi_- \ , \\
U^2(p,+1/2) &\rightarrow -\frac{i}{\sqrt{6}}\sqrt{p}\begin{pmatrix}1\\-1\end{pmatrix}\otimes\varphi_- \ , \\
U^3(p,+1/2) &\rightarrow \frac{1}{M}\sqrt{\frac{2}{3}}p^{3/2}\begin{pmatrix}1\\1\end{pmatrix}\otimes\varphi_+ \ .
\end{aligned} \tag{4.3}$$

(iii) For the $\chi$-spinors we have

$$\begin{aligned}
u_\chi(p,+1/2) &= \sqrt{3/2}\sqrt{4M}\begin{pmatrix}\cosh\frac{1}{2}\zeta\\ \sinh\frac{1}{2}\zeta\end{pmatrix}\otimes\varphi_+ \rightarrow \sqrt{3/2}\sqrt{p}\begin{pmatrix}1\\1\end{pmatrix}\otimes\varphi_+ \ , \\
u_\chi(p,-1/2) &= \sqrt{3/2}\sqrt{4M}\begin{pmatrix}\cosh\frac{1}{2}\zeta\\ -\sinh\frac{1}{2}\zeta\end{pmatrix}\otimes\varphi_- \rightarrow \sqrt{3/2}\sqrt{p}\begin{pmatrix}1\\-1\end{pmatrix}\otimes\varphi_- \ .
\end{aligned} \tag{4.4}$$

Notice the particular normalization of the $\chi$-spinors, which because of the E.T.A.C.-relations has a factor $\sqrt{3/2}$.

(iv) For the special solution we analyze the following cartesian components

$$\begin{aligned}
\hat{u}^0(p;+1/2) &= \frac{1}{3}\left(\gamma^0 - 2\frac{p_0}{M}\right)\ u_\chi(p,+1/2) \ , \\
\hat{u}^1(p;+1/2) &= \frac{1}{3}\left(\gamma^1 - 2\frac{p^1}{M}\right)\ u_\chi(p,+1/2) = \frac{1}{3}\gamma^1 u_\chi(p,+1/2) \ , \\
\hat{u}^2(p;+1/2) &= \frac{1}{3}\left(\gamma^2 - 2\frac{p^2}{M}\right)\ u_\chi(p,+1/2) = \frac{1}{3}\gamma^2 u_\chi(p,+1/2) \ , \\
\hat{u}^3(p;+1/2) &= \frac{1}{3}\left(\gamma^3 - 2\frac{p_3}{M}\right)\ u_\chi(p,+1/2) \ ,
\end{aligned} \tag{4.5}$$

Then, we obtain in the massless limit

$$\begin{aligned}
\hat{u}^0(p;+1/2) &\rightarrow \frac{1}{\sqrt{6}}\sqrt{p}\begin{pmatrix}1\\-1\end{pmatrix}\otimes\varphi_+ - \frac{1}{M}\sqrt{\frac{2}{3}}\ p^{3/2}\begin{pmatrix}1\\1\end{pmatrix}\otimes\varphi_+ \ , \\
\hat{u}^1(p;+1/2) &\rightarrow +\frac{1}{\sqrt{6}}\sqrt{p}\begin{pmatrix}1\\-1\end{pmatrix}\otimes\varphi_- \ , \\
\hat{u}^2(p;+1/2) &\rightarrow +\frac{i}{\sqrt{6}}\sqrt{p}\begin{pmatrix}1\\-1\end{pmatrix}\otimes\varphi_- \ , \\
\hat{u}^3(p;+1/2) &\rightarrow \frac{1}{\sqrt{6}}\sqrt{p}\begin{pmatrix}1\\-1\end{pmatrix}\otimes\varphi_+ - \frac{1}{M}\sqrt{\frac{2}{3}}\ p^{3/2}\begin{pmatrix}1\\1\end{pmatrix}\otimes\varphi_+ \ .
\end{aligned} \tag{4.6}$$

Next we use these for the analysis of the massless limit for [1]

$$u^\mu(p;+1/2) \;=\; U^\mu(p,+1/2) + \hat{u}^\mu(p;+1/2) \tag{4.10}$$

Using the results above, we find

$$\begin{aligned}
u^0(p;+1/2) &\;\rightarrow\; \frac{1}{\sqrt{6}}\sqrt{p}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \varphi_+ \ , \\
u^1(p;+1/2) &\;\rightarrow\; -\frac{1}{\sqrt{6}}\sqrt{p}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \varphi_- + \frac{1}{\sqrt{6}}\sqrt{p}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \varphi_- = 0 \ , \\
u^2(p;+1/2) &\;\rightarrow\; -\frac{i}{\sqrt{6}}\sqrt{p}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \varphi_- + \frac{i}{\sqrt{6}}\sqrt{p}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \varphi_- = 0 \ , \\
u^3(p;+1/2) &\;\rightarrow\; \frac{1}{\sqrt{6}}\sqrt{p}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \varphi_+ \ .
\end{aligned} \tag{4.11}$$

We notice that we can write this result in the form

$$u^\mu(p;+1/2) \rightarrow \frac{1}{\sqrt{6}}\,\frac{p^\mu}{\sqrt{p}}\,\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \varphi_+ \ . \tag{4.12}$$

Similarly, we can write

$$u^\mu(p;-1/2) \rightarrow \frac{1}{\sqrt{6}}\,\frac{p^\mu}{\sqrt{p}}\,\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \varphi_- \ . \tag{4.13}$$

---

[1] The addition of the two spinors in momentum space needs some comment. Starting from the Fourier spectral representation

$$\psi^\mu(x) = \int \frac{d^4p}{(2\pi)^4}\ \left[U^\mu(p)\delta(p^2 - M^2) + \hat{u}^\mu(p)\delta(p^2 - 4M^2)\right]\ e^{-ip\cdot x} \ . \tag{4.7}$$

the expression between brackets in the massless limit gives

$$\Big[\ldots\Big] \rightarrow \Big[U^\mu(p) + \hat{u}^\mu(p)\Big]_{M=0} \delta(p^2) + O(M) \ . \tag{4.8}$$

Here, we use that the spinors contain terms going at most like $O(1/M)$. Therefore, in the massless limit

$$\psi^\mu(x) \;\rightarrow\; \int \frac{d^4p}{(2\pi)^4}\ u^\mu(p)\ \delta(p^2)\ e^{-ip\cdot x} \ \ \text{with}\ \ u^\mu(p) \equiv U^\mu(p) + \hat{u}^\mu(p) \ . \tag{4.9}$$

## V. THE SPHERICAL BASIS MASSLESS SPINORS

(i) Spherical Rarita-Schwinger spinors

$$\begin{aligned}
U^0(p,+1/2) &\to \frac{1}{M}\sqrt{\frac{2}{3}}p^{3/2}\begin{pmatrix}1\\1\end{pmatrix}\otimes\varphi_+ \ ,\\
U^+(p,+1/2) &= \frac{1}{2}\left(U^1+iU^2\right)\to 0\ ,\\
U^-(p,+1/2) &= \frac{1}{2}\left(U^1-iU^2\right)\to -\frac{1}{\sqrt{6}}\sqrt{p}\begin{pmatrix}1\\-1\end{pmatrix}\otimes\varphi_- \ ,\\
U^3(p,+1/2) &\to \frac{1}{M}\sqrt{\frac{2}{3}}p^{3/2}\begin{pmatrix}1\\1\end{pmatrix}\otimes\varphi_+ \ .
\end{aligned} \tag{5.1}$$

(ii) For the spherical special solution spinors we analyze

$$\begin{aligned}
\hat{u}^0(p;1/2) &= \frac{1}{3}\left(\gamma^0-2\frac{p_0}{M}\right)\ u_\chi(p,+1/2)\ ,\\
\hat{u}^\pm(p;1/2) &= \frac{1}{3}\left(\gamma^\pm-2\frac{p^\pm}{M}\right)\ u_\chi(p,+1/2)=\frac{1}{3}\gamma^\pm u_\chi(p,+1/2)\ ,\\
\hat{u}^3(p;1/2) &= \frac{1}{3}\left(\gamma^3-2\frac{p_3}{M}\right)\ u_\chi(p,+1/2)\ ,
\end{aligned} \tag{5.2}$$

where we introduced

$$\gamma^\pm = \frac{1}{2}\left(\gamma^1\pm i\gamma^2\right)=\begin{pmatrix}0 & \sigma_\pm\\ -\sigma_\pm & 0\end{pmatrix}\ , \tag{5.3}$$

with $\sigma_\pm = (\sigma_1 \pm i\sigma_2)/2$. Also, $\epsilon^+(p,+1) = 0, \epsilon^+(p,-1) = 1/\sqrt{2}$, $\epsilon^-(p,+1) = -1/\sqrt{2}, \epsilon^-(p,-1) = 0$, and $\epsilon^\pm(p,0) = 0$.

Then, we obtain in the massless limit

$$\begin{aligned}
\hat{u}^0(p;+1/2) &\to \frac{1}{\sqrt{6}}\sqrt{p}\begin{pmatrix}1\\-1\end{pmatrix}\otimes\varphi_+ - \frac{1}{M}\sqrt{\frac{2}{3}}\ p^{3/2}\begin{pmatrix}1\\1\end{pmatrix}\otimes\varphi_+\ ,\\
\hat{u}^+(p;+1/2) &\to 0\ ,\\
\hat{u}^-(p;+1/2) &\to +\frac{1}{\sqrt{6}}\sqrt{p}\begin{pmatrix}1\\-1\end{pmatrix}\otimes\varphi_-\ ,\\
\hat{u}^3(p;+1/2) &\to \frac{1}{\sqrt{6}}\sqrt{p}\begin{pmatrix}1\\-1\end{pmatrix}\otimes\varphi_+ - \frac{1}{M}\sqrt{\frac{2}{3}}\ p^{3/2}\begin{pmatrix}1\\1\end{pmatrix}\otimes\varphi_+\ .
\end{aligned} \tag{5.4}$$

(iii) Next we use these for the analysis of the massless limit for

$$u^\mu(p;1/2) = U^\mu(p,+1/2)+\hat{u}^\mu(p;+1/2) \tag{5.5}$$

Using the results above, we find

$$\begin{aligned}
u^0(p;+1/2) &\rightarrow +\frac{1}{\sqrt{6}}\sqrt{p}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \varphi_+ \ , \\
u^+(p;+1/2) &\rightarrow 0 \\
u^-(p;+1/2) &\rightarrow 0 \\
u^3(p;+1/2) &\rightarrow +\frac{1}{\sqrt{6}}\sqrt{p}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \varphi_+ \ .
\end{aligned} \tag{5.6}$$

## VI. THE MASSLESS PROJECTION/PROPAGATOR

The massless progagator was found to be

$$\begin{aligned}
S_F^{\mu\nu}(p) &= \left[-\not{p}\left(g^{\mu\nu} - \frac{1}{2}\gamma^\mu\gamma^\nu\right) + \gamma^\mu p^\nu\right]\frac{1}{p^2+i\epsilon} - \not{p}\frac{2p^\mu p^\nu}{(p^2+i\epsilon)^2} \\
&= -\not{p}\left(g^{\mu\nu} - \frac{1}{2}\gamma^\mu\gamma^\nu\right)\frac{1}{p^2+i\epsilon} + \not{p}\frac{\not{p}\gamma^\mu p^\nu - 2p^\mu p^\nu}{(p^2+i\epsilon)^2} \\
&= -\left(g^{\mu\nu} - \frac{1}{2}\gamma^\mu\gamma^\nu\right)\frac{\not{p}}{p^2+i\epsilon} + \frac{p^\mu\gamma^\nu\not{p} - 2p^\mu p^\nu}{(p^2+i\epsilon)^2}\not{p} \ .
\end{aligned} \tag{6.1}$$

The $p^\mu p^\nu$-term is a dipole-ghost like term, and it is obvious that we can not reproduce this term from our spinor solutions. So, in the following we concentrate on the 'regular' part, i.e.

$$\hat{S}_F^{\mu\nu}(p) = \left[-\not{p}\left(g^{\mu\nu} - \frac{1}{2}\gamma^\mu\gamma^\nu\right) + \gamma^\mu p^\nu\right]\frac{1}{p^2+i\epsilon} \equiv \frac{\Lambda^{\mu\nu}(p)}{p^2+i\epsilon} \ . \tag{6.2}$$

Starting from

$$u^\mu(p;+1/2) = U^\mu(p,+1/2) + \hat{u}^\mu(p;+1/2) \tag{6.3}$$

we study the projection operator

$$\Lambda^{\mu\nu}(p) = \sum_{\lambda=-3/2}^{+3/2} u^\mu(p;\lambda)\bar{u}^\nu(p;\lambda) \ . \tag{6.4}$$

Working with $p^\mu = (p,0,0,p)$ we found the following spinors/components:

$$\begin{aligned}
u^\mu(p,+3/2) &\rightarrow \sqrt{p}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \varphi_+ \cdot \epsilon^\mu(+1) \ , \\
u^\mu(p,-3/2) &\rightarrow \sqrt{p}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \varphi_- \cdot \epsilon^\mu(-1) \ , \\
u^\mu(p;+1/2) &\rightarrow \frac{1}{\sqrt{6}}\,\frac{p^\mu}{\sqrt{p}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \varphi_+ \ , \\
u^\mu(p;-1/2) &\rightarrow \frac{1}{\sqrt{6}}\,\frac{p^\mu}{\sqrt{p}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \varphi_- \ .
\end{aligned} \tag{6.5}$$

1. $\Lambda^{00}(p)$: From the properties of the $\epsilon^\mu(\pm 1)$-vectors it follows that there is no contribution from the $\lambda = \pm 3/2$ helicities. From the helicities $\lambda = \pm 1/2$ we get

$$\begin{aligned}
\lambda = -1/2 \ : \ \Lambda^{00}(p) &\leftarrow \frac{p}{6}\begin{pmatrix} \varphi_- \\ \varphi_- \end{pmatrix}\begin{pmatrix} \varphi_-^\dagger & -\varphi_-^\dagger \end{pmatrix} \\
&= \frac{p}{6}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & -1 \end{pmatrix} = \frac{p}{6}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \\
\lambda = +1/2 \ : \ \Lambda^{00}(p) &\leftarrow \frac{p}{6}\begin{pmatrix} \varphi_+ \\ -\varphi_+ \end{pmatrix}\begin{pmatrix} \varphi_+^\dagger & \varphi+-^\dagger \end{pmatrix} \\
&= \frac{p}{6}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix} = \frac{p}{6}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ,
\end{aligned}$$

which upon addition gives that

$$\Lambda^{00}(p) \Rightarrow \frac{p}{6}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \Rightarrow \left[-\not{p}\left(g^{\mu\nu} - \frac{1}{2}\gamma^\mu\gamma^\nu\right) + \gamma^\mu p^\nu\right]/3 , \tag{6.6}$$

where the last identity is valid for $p^\mu = (p, 0, 0, p)$ and $\mu = \nu = 0$.

The discrepancy factor 3 can be removed by changing the normalization of the $\chi$-spinors by a factor $\sqrt{3}$.
2. $\Lambda^{33}(p)$: Also here, only the $\lambda = \pm 1/2$ will contribute. Since the $\mu = 0$ and $\mu = 3$ components of $u^\mu(p, \pm 1/2)$ are the same, we get for $\Lambda^{33}(p)$ the same result as for $\Lambda^{00}(p)$. This is also consistent with (6.2), which gives

$$\Lambda^{00}(p) = \Lambda^{33}(p) = \frac{1}{2}\left(\gamma^0 + \gamma^3\right) . \tag{6.7}$$

3. $\Lambda^{11}(p)$: In this case only $\lambda = \pm 3/2$ will contribute. A similar computation as for $\Lambda^{00}$ above gives the result

$$\Lambda^{11}(p) \Rightarrow \frac{p}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix} \Rightarrow \frac{1}{2}\not{p} = \frac{1}{2}p(\gamma^0 - \gamma^3) , \tag{6.8}$$

which is in full agreement with (6.2). Obviously, $\Lambda^{22}(p) = \Lambda^{11}(p)$.

4. $\Lambda^{03}(p), \Lambda^{30}(p)$: In equation (6.2) we have for these cases

$$\begin{aligned}
\Lambda^{03} &= \frac{1}{2}\not{p}\gamma^0\gamma^3 + \gamma^0 p^3 = \frac{p}{2}\left(\gamma^0 + \gamma^3\right) = \Lambda^{00}(p) , \\
\Lambda^{30} &= \frac{1}{2}\not{p}\gamma^3\gamma^0 + p^0\gamma^3 = \frac{p}{2}\left(\gamma^0 + \gamma^3\right) = \Lambda^{00}(p) .
\end{aligned} \tag{6.9}$$

This is no surprise since as the $u^\mu(p)(p,\pm 1/2)$ is concerned there is no difference between the zero'th and third component.

5. $\Lambda^{12}(p), \Lambda^{21}(p)$: From (6.2) we expect to find

$$\Lambda^{12}(p) = \frac{1}{2}\not{p}\gamma^1\gamma^2 = i\frac{p}{2}\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix} . \tag{6.10}$$

From the massless solutions we expect

$$\begin{aligned}
\Lambda^{12}(p) &= \sum_{\lambda=\pm 3/2} U^1(p,\lambda)\bar{U}^2(p,\lambda) \to p\begin{pmatrix} \varphi_+ \\ \varphi_+ \end{pmatrix}\begin{pmatrix} \varphi_+^\dagger & -\varphi_+^\dagger \end{pmatrix}\epsilon^1(+1)\epsilon^2(+1)^* \\
&+p\begin{pmatrix} \varphi_- \\ -\varphi_- \end{pmatrix}\begin{pmatrix} \varphi_-^\dagger & \varphi_-^\dagger \end{pmatrix}\epsilon^1(-1)\epsilon^2(-1)^* \\
&= -i\frac{p}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 & 0 \end{pmatrix} + i\frac{p}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix} \\
&= -i\frac{p}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + i\frac{p}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix} \\
&= i\frac{p}{2}\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix} \Leftarrow \frac{1}{2}\not{p}\gamma^1\gamma^2 .
\end{aligned} \tag{6.11}$$

which is indeed identical to (6.7). Obviously $\Lambda^{21}(p) = -\Lambda^{12}(p)$.
This concludes the analysis of all components of $\Lambda^{\mu\nu}(p)$ not equal to zero.

## VII. GAUGE-SYMMETRY AND THE DIPOLE-GHOST TERM

For $M = 0$ one has the 'gauge-symmetry ' (!?)

$$\psi^\mu(x) \to \psi^{\prime,\mu}(x) = \psi^\mu(x) + ic\partial^\mu\eta(x) , \tag{7.1}$$

where $\eta(x)$ is a massless spin-1/2 field (grassmann-variable), i.e. $\not{p}\eta(x) = 0$. The corresponding spinor, denoted by $\eta(p,\pm 1/2)$, adds to the projection operator the term

$$\begin{aligned}
\Lambda_\eta^{\mu\nu}(p) &= \sum_{\lambda=\pm 1/2} p^\mu\eta(p,\lambda)\bar{\eta}(p,\lambda)p^\nu = \frac{p}{2}\left\{\begin{pmatrix} \varphi_+ \\ \varphi_+ \end{pmatrix}\begin{pmatrix} \varphi_+^\dagger & -\varphi_+^\dagger \end{pmatrix} + \begin{pmatrix} \varphi_- \\ -\varphi_- \end{pmatrix}\begin{pmatrix} \varphi_-^\dagger & \varphi_-^\dagger \end{pmatrix}\right\}\cdot p^\mu p^\nu \\
&= \frac{p}{2}\left(\gamma^0 - \gamma^3\right)\ p^\mu p^\nu \Rightarrow \frac{1}{2}\not{p}\ p^\mu p^\nu .
\end{aligned} \tag{7.2}$$

Therefore, we can 'gauge away' the dipole-ghost term in the propagator (6.1) by choosing

$$c = \sqrt{\frac{2}{-\Box + i\epsilon}} \ , \tag{7.3}$$

which gives for the spinors of the $\eta^\mu(x) = ic\partial^\mu\eta(x)$ field the spinor

$$u^\mu_\eta(p,s) = \sqrt{\frac{2}{p^2 + i\epsilon}}\ p^\mu\eta(p,s)\ . \tag{7.4}$$

Then, the propagator for the gauged $\psi^\mu(x)$-field becomes

$$S'^{,\mu\nu}_F(p) = \left[-\not p\left(g^{\mu\nu} - \frac{1}{2}\gamma^\mu\gamma^\nu\right) + \gamma^\mu p^\nu\right]\frac{1}{p^2 + i\epsilon}\ , \tag{7.5}$$

i.e. a 'normal' propagator without dipole-ghost term. We can take out a $\not p$-factor and write (7.5) in the form

$$S'^{,\mu\nu}_F(p) = \not p\left[-\left(g^{\mu\nu} - \frac{1}{2}\gamma^\mu\gamma^\nu\right) + \frac{\not p\gamma^\mu p^\nu}{p^2 + i\epsilon}\right]\frac{1}{p^2 + i\epsilon} \equiv \frac{\Lambda^{\mu\nu}(p)}{p^2 + i\epsilon}\ , \tag{7.6}$$

## VIII. CONCLUSIONS

We found that the $u^\mu$-spinor consists of a massless purely spin-3/2 and a purely spin-1/2 part, i.e.

$$u^\mu(p;\lambda) = u^\mu_{3/2}(p,\lambda) + u^\mu_{1/2}(p;\lambda) \tag{8.1}$$

where for $u^\mu_{3/2}$ the helicity $\lambda = \pm 3/2$, and for for $u^\mu_{1/2}$ the helicity $\lambda = \pm 1/2$. Moreover, since the spin-1/2 spinor is proportional to $p^\mu$ the spin-1/2 part will decouple from a concerved current with $\partial^\mu J_\mu = 0$. Last but not least, the dipole-ghost term in the massless propagator can be 'gauged' away.

## APPENDIX A: GROUP THEORETICAL INTERMEZZO

In this appendix we describe the relation between the little groups $SO(3)$ and $E2$ which are the invariance groups for the four-vectors $\overset{o}{p}{}^\mu = (p^0 = M, 0, 0, 0)$ respectively $\overset{o}{p}{}^\mu = (p, 0, 0, p)$. The first denotes the 4-momentum of a particle of mass M in rest, and the second one the 4-momentum of a massless particle with $p^2 = 0$. To connect these two cases, we consider the Lie-algebra pertinent to the 4-vector $p^\mu = (p^0, 0, 0, p)$. A basis for this Lie-algebra if given by

$$L_1 \ = \ J_1 + \tanh\chi_p\ K_2 \quad , \quad L_2 = J_2 - \tanh\chi_p\ K_1 \quad , \quad J_3\ , \tag{A1}$$

where $\cosh\chi_p = p^0/M, \sinh\chi_p = p/M$. The elements $K_1, K_2$ are the generators of the special Lorentz transformation along the x-, respectively the y-axis, and $J_3$ for the rotations in the xy-plane. The Lie-algebra for the Lorentz group are

$$[J_i, J_j] \ = \ i\epsilon_{ijk}\ J_k\ , \tag{A2a}$$
$$[J_i, K_j] \ = \ i\epsilon_{ijk}\ K_k\ , \tag{A2b}$$
$$[K_i, K_j] \ = \ i\epsilon_{ijk}\ J_k\ . \tag{A2c}$$

Using this algebra, we derive that

$$
\begin{aligned}
[J_3, L_1] &= [J_3, J_1] + \tanh\chi_p\,[J_3, K_2] \\
&= i\,(J_2 - \tanh\chi_p K_1) = iL_2\ , && \text{(A3a)} \\
[J_3, L_2] &= [J_3, J_2] - \tanh\chi_p\,[J_3, K_1] \\
&= -i\,(J_1 + \tanh\chi_p K_2) = -iL_1\ , && \text{(A3b)} \\
[L_1, L_2] &= [J_1, J_2] - \tanh^2\chi_p\,[K_2, K_1] \\
&= i\cosh^{-2}\chi_p J_3 = i\frac{M^2}{p^2 + M^2}\ J_3\ . && \text{(A3c)}
\end{aligned}
$$

We find from this algebra:

1. <u>Massive case</u>: For the particle at rest $\mathbf{p} = 0$ and taking as a basis the elements

$$A_1 = \cosh\chi_p\ L_1\quad,\quad A_2 = \cosh\chi_p\ L_2\quad,\quad A_3 = J_3\ , \tag{A4}$$

which for $p = 0$ satify the Lie-algebra isomorphic to $SO(3)$:

$$[A_i, A_j] = i\epsilon_{ijk}\ A_k\ . \tag{A5}$$

2. <u>Massless case</u>: For a massless particle $M = 0$, and the algebra in (A3c) reduces to a Lie-algebra isomorphic to the Euclidean group in two dimensions $E_2$

$$[L_1, J_3] = -iL_2\quad,\quad [L_2, J_3] = +iL_1\quad,\quad [L_1, L_2] = 0\ . \tag{A6}$$

The consequence of the abelian subalgebra, spanned by $(L_1, L_2)$, is that the helicities $\lambda$ for massless particles of spin j can only assume the values $\lambda = \pm j$.

[1] J. W. Wagenaar, *Pion-Nucleon Scattering in Kadyshevsky Formalism and Higher Spin Field Quantization*, Ph.D. thesis University of Nijmegen, July 2009, e-Print: arXiv:0904.1398[nucl-th].
[2] J. W. Wagenaar and T.A. Rijken, *On the Quantization of the Higher Spin Fields*, e-Print: arXiv:0905.3893[hep-th].
[3] S. Weinberg, Phys. Rev. **133B**, 1318 (1964); *Feynman rules for any spin*, Lecture notes on Particles and Field Theory, Brandeis Summer Institute in Theoretical Physics (1964), Prentice-Hall Englewood Cliffs, N.J. (1965).