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Massive Spin-2: Field-equations, Propagators, Massless-limit, and Perihelium Precessions

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Abstract

Background: This paper presents the quantization of massive and massless spin-2 particles using the auxiliary-field method. The issue of the perihelium precessions for a massive graviton in agreement with the data is studied in the context of this spin-2 theory in tree-approximation.

Purpose: The aim is to study the masless limit and to investigate the perihelium of the planets as a function of the graviton mass, and to calculate the effects to first order in the graviton mass.

Method: The field theory for the spin-2 particles is constructed using the Dirac quantization method. In order to impose sufficient constraints on the spin-2 tensor $h^{\mu\nu}(x)$-field, an auxiliary vector-ghost field $\eta^\mu(x)$ and a complex scalar-ghost field $\epsilon(x)$ are introduced. The $h^{\mu\nu}(x)$-field is coupled to a conserved energy-momentum tensor, which results in a dependence of the $h^{\mu\nu}$-propagator on the $\epsilon(x)$-field for the massive case.

The gravitational interaction between the sun and the planets (treated as scalar particles) is introduced as in the weak-field approximation in general relativity, *i.e.* by a coupling of the $h^{\mu\nu}(x)$-field to the matter energy-momentum tensors.

Results: A general Lagrangian, containing parameters A, $\mathcal{M}$, and b_2, for the massive spin-2 (tensor) formalism using the auxiliary spin-1 (vector) and spin-0 (scalar) fields is reviewed. It is found that only A=-1 leads to a physical theory. Furthermore, it appeared that for the proper transition to a massless spin-2 theory the limit $b_2 \sim b \to \pm\infty$ is required. By making a suitable field-transformation, a theory is obtained for massive and massless spin-2 fields with an imaginary-scalar-ghost $\epsilon(x)$ and a vector-ghost $\rho^\mu(x)$ field, both satisfying free Klein-Gordon equations. Furthermore, it is shown that the ρ-field is eliminated from this model for $b = \pm\infty$. The quantization of the ϵ ghost-field is analyzed. Using a standard Gupta-condition for physical states, taking care of the $\epsilon(x)$ ghost-states in the standard manner, a massive spin-2 quantum field theory with a spin-0 scalar-ghost is reached.

Coupling the $h^{\mu\nu}(x)$-field to the energy-momentum tensor for the sun and planets, the non-newtonian correction to the perihelium precession is in accordance with the Einstein result also for a non-zero graviton mass.

Conclusions: In the context of this setting it is found that, in tree-approximation, a small spin-2 graviton mass is compatible with the perihelium-precession of Mercury etc.

I. INTRODUCTION

It is one of the aims of this investigation to find a massive quantized spin-2 theory in Minkowski-space, henceforth refered to as a linearized relativistic gravitation theory with axialiary fields (LRGT-AF). In tree-approximation, LRGT-AF allows (i) a smooth massless limit, and (ii) a perturbative expansion in the small ("graviton") mass M. Here, we exploit the auxiliary field (AF) method with a vector and a scalar field. [1]. For the latter to be meaningful, it is necessary that the theory satisfies the following requirements: (i) unitarity, and (ii) a correct massless limit. This would open the possibility of giving a small mass to the graviton without destroying e.g. the correct prediction for the perihelium of Mercury. In the literature this issue has been discussed both in Minkowski [1, 2] and in de Sitter space [3–5].
It appears that this is impossible without making use of (complex-)ghost fields. The possibility to exploit complex ghost-fields is analyzed in detail in this paper using methods discussed in the literature, see [6]. By making a simple field-transformation the $\eta_\mu(x)$ and $\epsilon(x)$ fields can be decoupled and a model is obtained for massive spin-2 field with free auxiliary fields $\rho_\mu(x)$ and $\epsilon(x)$. Using Gupta conditions for the latter a quantum field theory with a complex-ghost field is reached. Henceforth we refer to this formalism as the auxiliary-field quantum field theory (LRGT-AF). This form of the massive spin-2 formalism is similar to the so-called "B-field formalism" for QED,QCD and massive vector-mesons as described in Refs. [7, 8].
In the LRGT-AF-formalism the massless limit can be studied in detail, and it is shown that in this formalism a small spin-2 (graviton) mass is compatible with the Einstein non-newtonian correction of the perihelium-precession of Mercury etc.. From recent observations [9–11] and studies [12] the upper limit for the graviton mass seems to be $\mu_G \leq 2\,10^{-38}\, m_e = 18.22\times 10^{-66}$ g, and is estimated in these notes to give a really tiny correction to the perihelium precession of Mercury.

The contents of this paper consists of three parts. In the first part the attention is focussed on the field equations, the quantization with the Dirac-method, the commutations relations, and the Feynman-propagator. In the second part (i) the (causal) quantization of the complex auxiliary ghost-field is reviewed, and (ii) the final field theory model is formulated which is designed for the computation of the perihelium precession, hving a smooth massless limit. The precession of Mercury is calculated with a finite mass-correction for the "graviton". The third part contains a number of appendices containing supporting material.

First part: In section II the general Lagrangian for the spin-2 $h^{\mu\nu}(x)$, the auxiliary spin-1 $\eta^\mu(x)$, and spin-0 $\epsilon(x)$ fields is given, which contains the graviton-mass M_2-, the scale-mass $\mathcal{M}$-, the A and b-parameter. Here also the field equations are derived. Furthermore, the decoupling of the vector and scalar auxiliary field is achieved via a field transformation. Here also the coupled Klein-Gordon equations for the spin-2, the spin-1, and spin-0 fields are given. In section III the Dirac's Hamiltonian method, appropriate for the quantization of constrained systems, is reviewed for the spin-2 tensor-field exploying the auxiliary field method with a vector and scalar filed. The canonical momenta are defined, the Hamiltonian is given, the canonical momenta and the equal-time commutation (ETC) relations are listed. In section IV an integral representation for solutions of the free Klein-Gordon equations for the tensor, vector, and scalar fields is used to obtain the non-equal-time commutation (NETC)

[1] In sections I-VIII the graviton mass is notated by M_2 or M and after that by μ_G.

relations. This path also leads to the Feynman-propagator for the tensor-field. Section V is devoted to the question whether a representation for the spin-2 propagator etc. can be found in the "*b-parameter space*" that (i) allows a smooth massless limit, and (ii) such that for $M \neq 0$ the theory contains besides the spin-2 propagator also a physical acceptable spin-0 propagator. This leads to $b \to \pm\infty$. We summarize our preliminary conclusions w.r.t. the impossibility of a massive spin-2 theory with a no-ghost scalar field and a correct prediction for the perihelia.
Section VI (i) It is shown that the ρ-field can be eliminated from the model for $b \to \pm\infty$, leaving only the auxiliary (ghost) ϵ-field. In section VII the symptotic fields are introduced and the physical contents of the $h_{\mu\nu}(x)$-fields is analyzed for the massive and massless case. Spin-2 $U_{\mu\nu}(x)$-fields are introduced and it is shown explicitly that there is a smooth transition from the massive to the massless spin-2 field formalism. This is the advantage of the auxiliary-field method as compared to the Proca-method [8].

Second part: In section VIII the massive spin-2 model of this paper is applied to a computation of the non-newtonian correction of the perihelium-precession of Mercury. The gravitational interaction between the sun and planets is introduced via the coupling of the $h^{\mu\nu}(x)$ spin-2 field to the energy-momentum tensors. In section IX the contribution to the perihelium precession is computed for the scalar and scalar-ghost part of the tensor-field propagator. The results on the perihelium precession are summarized and compared with solar-system data. The finite-mass correction are computed and shown to be negligible. In section XI we discuss the results and compare the LRGT-AF with other models in the literature.

Appendices: In Appendix A the (causal) quantization of purely-imaginary $\epsilon(x)$-field is treated, and the propagator is derived in detail. In Appendix B the smooth limit of the model is analyzed considering the decoupling of the "false helicities" $\lambda \neq \pm 2$. In Appendix C starting with the Bethe-salpeter equation for two-scalar particles, *e.g.* the sun and planet, the local and non-local potentials for the Schrödinger equation are derived for the planetary motion. In Appendix D the matrix element of the scalar particle interaction for imaginary-ghost exchange is worked out in detail. In Appendix E the gravitational cross-term field energy for the planet-sun sytem, giving a -1/6 correction to the perihelium precession, is expicitly evaluated. In Appendix F the cosmological term is incorperated in the spin-2 formalism in the weak field approximation. The possible relation between the cosmological constant and graviton mass is discussed and consequences for cosmological parameters are estimated. Finally, Appendix G contains the derivation of $\sqrt{-g}$ up to second order in the $h^{\mu\nu}$-field.

II. MASSIVE GRAVITATION FIELD, EULER-LAGRANGE EQUATIONS

In the work on the quantization of the spin-2 fields [13], are used the symmetric $h_{\mu\nu}$-tensor field, and the two auxiliary (ghost-)fields $\eta^\mu(x)$ and $\epsilon(x)$. The Lagrangian consists of three parts $\mathcal{L}_{2,\eta\epsilon} = \mathcal{L}_2 + \mathcal{L}_{GF} + \mathcal{L}_{int}$ which are specified below. In [15] the most general $\mathcal{L}_2$ is parametrized in terms of the parameters A,B, and C, with $B = (3A^2 + 3A + 1)/2$ and $C = 3A^2 + 3A + 1$ [2] where, see [13],

$$\begin{aligned}\mathcal{L}_2 &= \frac{1}{4}\partial^\alpha h^{\mu\nu}\partial_\alpha h_{\mu\nu} - \frac{1}{2}\partial_\mu h^{\mu\nu}\partial^\alpha h_{\alpha\nu} - \frac{1}{4}B\partial_\nu h^\beta_\beta \partial^\nu h^\alpha_\alpha - \frac{1}{2}A\partial_\alpha h^{\alpha\beta}\partial_\beta h^\nu_\nu \\ &\quad -\frac{1}{4}M_2^2 h^{\mu\nu}h_{\mu\nu} + \frac{1}{4}CM_2^2 h^\mu_\mu h^\nu_\nu, \end{aligned}\tag{2.1}$$

and application of the variation principle via the Euler-Lagrange ($\mathcal{E}.\mathcal{L}.$) equations gives the equation of motion (EoM) for the spin-2 field as well the constraints, for $A \neq -1/2$, *i.e.* analogous as the Proca-formalism for spin-1. However, in the Proca formalism $M_2 \neq 0$ is essential and hence prevents a useful massless limit [8]. Since for our purpose the massless limit is essential we use auxiliary fields, henceforth referred to as the B-field method.
For the symmetric $h_{\mu\nu}$-tensor field and the auxiliary $\eta^\mu(x)$ and $\epsilon(x)$ fields, the Lagrangian is $\mathcal{L}_{\eta\epsilon} = \mathcal{L}_2 + \mathcal{L}_{GF} + \mathcal{L}_{int}$ where [3]

$$\begin{aligned}\mathcal{L}_2 &= \frac{1}{4}\partial^\alpha h^{\mu\nu}\partial_\alpha h_{\mu\nu} - \frac{1}{2}\partial^\mu h^{\mu\nu}\partial^\alpha h_{\alpha\nu} - \frac{1}{4}B\,\partial_\nu h^\beta_\beta \partial^\nu h^\alpha_\alpha - \frac{1}{2}A\,\partial_\alpha h^{\alpha\beta}\partial_\beta h^\nu_\nu \\ &\quad -\frac{1}{4}M_2^2 h^{\mu\nu}h_{\mu\nu} + \frac{1}{16}M_2^2 h^\mu_\mu h^\nu_\nu, \end{aligned}\tag{2.2a}$$

$$\mathcal{L}_{GF} = \mathcal{M}\partial_\mu h^{\mu\nu}\eta_\nu + \mathcal{M}^2 h^\mu_\mu \epsilon + \frac{1}{2}b_2 M_2^2 \eta^\mu\eta_\mu, \tag{2.2b}$$

$$\mathcal{L}_{int} = \kappa\, h^{\mu\nu}\big(t_{M,\mu\nu} + t_{g,\mu\nu}\big). \tag{2.2c}$$

Here, we have introduced the scaling mass $\mathcal{M}$. Since we will investigate the massless limit $M_2 \to 0$ it will be convenient to distinguish this from the 'dynamical" spin-2 mass M_2. In the mass term for the η^μ-field we kept M_2, but the η^μ-mass is distinct from the spin-2 mass, namely $M_\eta^2 = -b_2 M_2^2$. [4] The purpose is to derive the spin-2 propagator via the quantization of the $h^{\mu\nu}(x)$-field. For that we need the constraints which follow from the "free field equations". So, in this section we take $\kappa = 0$.

First we work out the terms in the Euler-Lagrange equation (2.1)

$$\partial_\alpha \frac{\partial \mathcal{L}}{\partial_\alpha(h_{\mu\nu}(x))} - \frac{\partial\mathcal{L}}{\partial h_{\mu\nu}(x)} = 0, \tag{2.3}$$

and similarly for the $\eta^\mu(x)$ and $\epsilon(x)$ fields. This leads to the equations [13]

$$\begin{aligned}&\big(g^{\mu\alpha}g^{\nu\beta}\Box - 2\partial^\mu\partial^\alpha g^{\nu\beta} - B\,g^{\mu\nu}g^{\alpha\beta}\Box - A\,g^{\mu\nu}\partial^\alpha\partial^\beta + g^{\mu\alpha}g^{\nu\beta}M_2^2 \\ &-C\,g^{\mu\nu}g^{\alpha\beta}M_2^2\big)\,h_{\alpha\beta}(x) + \mathcal{M}\,(\partial^\mu\eta^\nu + \partial^\nu\eta^\mu)(x) - 2\mathcal{M}^2 g^{\mu\nu}\,\epsilon(x) = 0, \end{aligned}\tag{2.4a}$$

$$h^\mu_\mu = 0\ ,\ \ b_2 M_2^2\,\eta^\nu(x) + \mathcal{M}\partial_\mu h^{\mu\nu}(x) = 0. \tag{2.4b}$$

[2] These relations between A,B,C are connected to the constraints $h^\mu_\mu = 0$ and $\partial_\mu h^{\mu\nu} = 0$.
[3] In the following, we often denote the mass by $M_2 \equiv M$.
[4] In studying the limits $M_2 \to 0, b_2 \to \infty$ we keep $b_2 M_2^2$ fixed. **Later on in this appendix it will appear that for a proper description of the tensor-field commutators, we need to put** $\mathcal{M} = M_2$. This implies that we use here the same Lagrangian as in [13, 14].

Using $h^\alpha_\alpha = 0$ gives

$$\begin{aligned}\left(\Box + M_2^2\right)\ h^{\mu\nu}(x) &= 2\partial^\mu(\partial_\alpha h^{\alpha\nu}(x)) + A\, g^{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta}(x)\\ &\quad -\mathcal{M}\left(\partial^\mu\eta^\nu + \partial^\nu\eta^\mu\right)(x) + 2\mathcal{M}^2 g^{\mu\nu}\,\epsilon(x), \qquad (2.5a)\\ h^\mu_\mu &= 0\ ,\ \ \partial_\mu h^{\mu\nu}(x) = -\left(\frac{b_2 M_2^2}{\mathcal{M}}\right)\eta^\nu(x). \qquad (2.5b)\end{aligned}$$

With the last equation for $\partial_\alpha h^{\alpha\nu}(x)$ gives for $h^{\mu\nu}(x)$:

$$\begin{aligned}\left(\Box + M_2^2\right)\ h^{\mu\nu}(x) &= -\mathcal{M}\left(1 + b_2\frac{M_2^2}{\mathcal{M}^2}\right)\left(\partial^\mu\eta^\nu + \partial^\nu\eta^\mu\right)(x)\\ &\quad -g^{\mu\nu}\left(A\, b_2\frac{M_2^2}{\mathcal{M}}(\partial\cdot\eta(x)) - 2\mathcal{M}^2\epsilon(x)\right).\end{aligned} \tag{2.6}$$

Using $h^\mu_\mu = 0$ in the last equation above again, we obtain

$$0 = -2\mathcal{M}\left(1 + b_2\frac{M_2^2}{\mathcal{M}^2}\right)\partial\cdot\eta(x) - 4Ab_2\frac{M_2^2}{\mathcal{M}}\partial\cdot\eta(x) + 8\mathcal{M}^2\,\epsilon(x),$$

which gives the relation

$$\partial\cdot\eta(x) = 4\mathcal{M}\left[1 + (2A+1)b_2\frac{M_2^2}{\mathcal{M}^2}\right]^{-1}\epsilon(x). \tag{2.7}$$

Finally, we now substitute this relation into the field-equation for $h^{\mu\nu}$ and obtain

$$\begin{aligned}\left(\Box + M_2^2\right)\ h^{\mu\nu}(x) &= -\mathcal{M}\left(1 + b_2\frac{M_2^2}{\mathcal{M}^2}\right)\left(\partial^\mu\eta^\nu + \partial^\nu\eta^\mu\right)(x)\\ +2\mathcal{M}^2\left(1 + \frac{b_2M_2^2}{\mathcal{M}^2}\right)&\left[1 + (2A+1)\frac{b_2M_2^2}{\mathcal{M}^2}\right]^{-1} g^{\mu\nu}\,\epsilon(x).\end{aligned} \tag{2.8}$$

Next we derive the field-equations for $\eta^\mu(x)$ and $\epsilon(x)$. For that purpose we introduce the abbrevation [5]

$$b \equiv b_2\,(M_2^2/\mathcal{M}^2), \tag{2.9}$$

and the equations have the form

$$\begin{aligned}\left(\Box + M_2^2\right)\ h^{\mu\nu}(x) &= -\mathcal{M}(1+b)\left(\partial^\mu\eta^\nu + \partial^\nu\eta^\mu\right)(x)\\ &\quad +2\mathcal{M}^2(1+b)\big[1 + (2A+1)\,b\big]^{-1} g^{\mu\nu}\,\epsilon(x), \qquad (2.10a)\\ h^\mu_\mu(x) &= 0,\ \ \partial_\mu h^{\mu\nu}(x) = -b\mathcal{M}\eta^\nu,\ \ \partial\cdot\eta = 4\mathcal{M}\big[1 + (2A+1)\,b\big]^{-1}\epsilon(x) \qquad (2.10b)\end{aligned}$$

For the derivation for η^ν we start with $\partial_\mu h^{\mu\nu} = -b\mathcal{M}\eta^\nu$ which gives the relation

$$\partial_\mu(\Box + M_2^2)h^{\mu\nu} = -b\mathcal{M}(\Box + M_2^2)\eta^\nu \tag{2.11}$$

[5] In the following we will assume that $\mathcal{M} = M_2$. In the formulas we still use occasionally $\mathcal{M}$ and M_2.

Using the field-equation for $h^{\mu\nu}$ on the l.h.s. we get

$$\begin{aligned}\partial_\mu(\Box + M_2^2)h^{\mu\nu} &= -\mathcal{M}(1+b)\,(\Box\eta^\nu + \partial^\nu\partial\cdot\eta) + 2\mathcal{M}^2\,(1+b)\big[1+(2A+1)b\big]^{-1}\partial^\nu\epsilon(x) \\ &= -\mathcal{M}(1+b)\Box\eta^\nu - 2\mathcal{M}^2(1+b)\,[1+(2A+1)b]^{-1}\,\partial^\nu\epsilon(x) \qquad (2.12)\end{aligned}$$

The combination of (2.11) and (2.12) gives the η^ν-equation:

$$\left(\Box + M_\eta^2\right)\eta^\nu(x) = -2\mathcal{M}\frac{1+b}{1+(2A+1)b}\partial^\nu\epsilon(x) \;\text{ with } M_\eta^2 = -bM_2^2. \qquad (2.13)$$

The equation for $\epsilon(x)$ is obtained by differentiation of (2.13) and using (2.7) which leads to

$$\left(\Box + M_\epsilon^2\right)\epsilon(x) = 0 \;\text{ with }\; M_\epsilon^2 = -\frac{2b}{3+b}M_2^2. \qquad (2.14)$$

<u>Decoupling vector and scalar field:</u> Making the transformation $\eta^\mu \to \rho^\mu + \partial^\mu\Lambda$ with

$$\Lambda(x) \;=\; 2\frac{\mathcal{M}}{M_2^2}\frac{(3+b)}{b\,[1+(2A+1)b]}\epsilon(x), \qquad (2.15)$$

we arrive at the equations

$$\left(\Box - bM_2^2\right)\rho(x) = 0,\; \partial\cdot\rho(x) = 0, \qquad (2.16a)$$

$$\begin{aligned}&\left(\Box + M_2^2\right)h^{\mu\nu}(x) = -\mathcal{M}(1+b)\,(\partial^\mu\rho^\nu + \partial^\nu\rho^\mu)\,(x) \\ &+2\mathcal{M}^2(1+b)[1+(2A+1)b]^{-1}\left[g^{\mu\nu} - 2\frac{3+b}{b}\frac{\partial^\mu\partial^\nu}{M_2^2}\right]\epsilon(x). \qquad (2.16b)\end{aligned}$$

Summary of the field-equations:

$$\begin{aligned}&\left(\Box + M_2^2\right)h^{\mu\nu}(x) = -\mathcal{M}(1+b)\,(\partial^\mu\rho^\nu + \partial^\nu\rho^\mu)\,(x) \\ &+2\mathcal{M}^2(1+b)[1+(2A+1)b]^{-1}\left[g^{\mu\nu} - 2\frac{3+b}{b}\frac{\partial^\mu\partial^\nu}{M_2^2}\right]\epsilon(x), \qquad (2.17a)\end{aligned}$$

$$\left(\Box - bM_2^2\right)\rho(x) = 0, \qquad (2.17b)$$

$$\left(\Box - \frac{2b}{3+b}M_2^2\right)\epsilon(x) = 0, \qquad (2.17c)$$

and the constraints:

$$h^\mu_\mu(x) \;=\; 0,\;\; \partial\cdot\rho(x) = 0, \qquad (2.18a)$$

$$\partial_\mu h^{\mu\nu}(x) \;=\; -b\mathcal{M}\left(\rho^\nu + \frac{2\mathcal{M}}{M_2^2}\frac{3+b}{b(1-b)}\partial^\nu\epsilon\right)(x). \qquad (2.18b)$$

The vector and scalar auxiliary fields $\rho^\mu(x)$ and $\epsilon(x)$ are free (ghost) fields with an imaginary mass.

From equations (2.15) and (2.16) one easily derives for the "free" $h^{\mu\nu}$-field, that

$$(\Box + M_\rho^2)\rho^\mu = 0, \qquad (2.19a)$$

$$(\Box + M_\epsilon^2)(\Box + M_\rho^2)(\Box + M_2^2)h^{\mu\nu} = 0. \qquad (2.19b)$$

III. QUANTIZATION

Dirac's Hamiltonian method is appropriate for the quantization of constrained systems [13, 17]. The canonical momenta are defined as

$$\pi^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\dot{h}_{\mu\nu}}\ ,\ \pi^{\mu} = \frac{\partial\mathcal{L}}{\partial\dot{\eta}_{\mu}}\ ,\ \pi^{\epsilon} = \frac{\partial\mathcal{L}}{\partial\dot{\epsilon}}. \tag{3.1}$$

For the general Lagrangian $\mathcal{L}_{2,\eta\epsilon}$ one obtains

$$\begin{aligned}
\pi_{\mu\nu}(x) &= \frac{1}{2}\partial_0 h_{\mu\nu}(x) - \delta_{\mu 0}\partial^{\alpha} h_{\alpha\nu}(x) - \frac{1}{2}B\,\partial_0 h(x)\,\eta_{\mu\nu} \\
&\quad -\frac{1}{2}A\left(\delta_{\mu 0}\partial_{\nu} h(x) + \partial_{\alpha} h^{\alpha 0}(x)\,\eta_{\mu\nu}\right) + \mathcal{M}\delta_{\mu 0}\eta_{\nu}, && (3.2a)\\
\pi_{\eta,\nu} &= 0\ ,\ \pi_{\epsilon} = 0. && (3.2b)
\end{aligned}$$

Explicitly these momenta are [6]

$$\begin{aligned}
\pi_2^{00}(x) &= -\frac{1}{2}(1+2A+B)\,\dot{h}^{00}(x) - (1+\frac{1}{2}A)\partial_n h^{n0}(x) - \frac{1}{2}(A+B)\dot{h}^n_n(x), && (3.3a)\\
\pi_2^{0m}(x) &= +\frac{1}{2}\dot{h}^{0m}(x) - \partial_n h^{nm}(x) - \frac{1}{2}A\,\partial^m\left(h^{00}(x) + h^n_n(x)\right), && (3.3b)\\
\pi_2^{nm}(x) &= \frac{1}{2}\dot{h}^{nm}(x) - \frac{1}{2}B\left(\dot{h}^{00}(x) + \dot{h}^k_k(x)\right)\,\eta^{nm} - \frac{1}{2}A\left(\partial_k h^{k0}(x) + \dot{h}^{00}(x)\right)\eta^{mn}, && (3.3c)\\
\pi_{2k}^{\ k}(x) &= +\frac{1}{2}(1-3B)\,\dot{h}^k_k(x) - \frac{3}{2}(A+B)\,\dot{h}^{00}(x) - \frac{3}{2}A\,\partial_k h^{k0}(x). && (3.3d)
\end{aligned}$$

From the constraint $\partial_\mu h^{\mu\nu} = 0 (\mathcal{M} = 0)$ it follows that $\dot{h}^{0m} = -\partial_n h^{0n}$ and this gives

$$\pi_2^{0m}(x) = -\frac{1}{2}\partial_k h^{0k}(x) - \partial_n h^{nm}(x) - \frac{1}{2}A\,\partial^m\left(h^{00}(x) + h^n_n(x)\right),$$

i.e. a constraint and not an independent canonical-momentum variable. [7]
With $B = (3A^2 + 2A + 1)/2$ the momenta in (3.3) become

$$\begin{aligned}
\pi_2^{00}(x) &= -\frac{3}{4}(A+1)^2\,\dot{h}^{00}(x) - \frac{1}{2}(A+2)\partial_n h^{n0}(x) - \frac{1}{4}(3A+1)(A+1)\dot{h}^n_n(x), && (3.4a)\\
\pi_2^{0m}(x) &= +\frac{1}{2}\dot{h}^{0m}(x) - \partial_n h^{nm}(x) - \frac{1}{2}A\,\partial^m\left(h^{00}(x) + h^n_n(x)\right), && (3.4b)\\
\pi_2^{nm}(x) &= \frac{1}{2}\dot{h}^{nm}(x) - \frac{1}{4}(3A+1)(A+1)\,\dot{h}^{00}(x)\eta^{nm} - \frac{1}{4}(3A^2+2A+1)\,\dot{h}^k_k(x)\,\eta^{nm} \\
&\quad -\frac{1}{2}A\,\partial_k h^{k0}(x)\,\eta^{mn}, && (3.4c)\\
\pi_{2k}^{\ k}(x) &= -\frac{1}{4}(3A+1)^2\,\dot{h}^k_k(x) - \frac{3}{4}(3A+1)(A+1)\,\dot{h}^{00}(x) - \frac{3}{2}A\,\partial_k h^{k0}(x). && (3.4d)
\end{aligned}$$

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[6] Note that for A=-1, B=C=1, π_2^{00} etc. agree with [13]. The π_2^{0m} leads to a constraint because of $\partial_\mu h^{\mu\nu}|f\rangle = 0$.

[7] **Remark:** In the Dirac-method of quantization the equations from the $\mathcal{E.L.}$ equations are always valid and can be used, for example for the time dependeces etc, for example leading to new constraints [17].

Trying to solve for $\dot{h}^{00}$ and $\dot{h}^n_n$ leads to the equation

$$\begin{pmatrix} \pi_2^{00}+\frac{1}{2}(A+2)\partial_n h^{n0} \\ \pi_{2k}^{\ k}+\frac{3}{2}A\partial_k h^{k0} \end{pmatrix} = \begin{pmatrix} -\frac{3}{4}(A+1)^2 & -\frac{1}{4}(3A+1)(A+1) \\ -\frac{3}{4}(3A+1)(A+1) & -\frac{1}{4}(3A+1)^2 \end{pmatrix} \begin{pmatrix} \dot{h}^{00} \\ \dot{h}^k_k \end{pmatrix}$$

Obviously, the determinant is zero. There are two solutions: A=-1 and A=-1/3. The case A=-1 leads to the velocity $\dot{h}^k_k = -\pi_{2k}^{\ k} + 3\partial_k h^{k0}/2$ and the constraint $\theta^{00} = \pi^{00} + \partial_n h^{n0}/2 = 0$.
The case A=-1/3 leads to the velocity $\dot{h}^{00} = -3\pi_2^{00} - 5\partial_n h^{n0}/2$ and the constraint $\theta_{2k}^{\ k} = \pi_{2k}^{\ k} - \partial_k h^{k0} = 0$. For gravity this solution is unphysical. Therefore, henceforth we will only consider the case A=-1, which is treated in Ref. [13].
The velocities are

$$\begin{aligned} \dot{h}^{nm} &= 2\pi_2^{nm} - \eta^{nm}\pi_{2k}^{\ k} + \frac{1}{2}\eta^{nm}\partial_k h^{k0}, && \text{(3.5a)} \\ \dot{h}_{2k}^{\ k} &= -\pi_{2k}^{\ k} + \frac{3}{2}\partial_k h^{k0}, && \text{(3.5b)} \end{aligned}$$

and the *primary* constraints

$$\begin{aligned} \theta_2^{00} &= \pi_2^{00} + \frac{1}{2}\partial_n h^{n0} - M_2\eta^0\,, \quad && \theta^0_\eta = \pi^0_\eta\,, \\ \theta_2^{0m} &= \pi_2^{0m} + \partial_n h^{nm} - \frac{1}{2}\partial^m h^{00}\,, && \theta^m_\eta = \pi^m_\eta\,, \\ & -\frac{1}{2}\partial^m h^n_n - M_2\eta^m\,, && \theta_\epsilon = \pi_\epsilon\,. \end{aligned} \tag{3.6}$$

which vanish in the *weak* sense [17, 18].

A. Dirac-theory: The Hamiltonian and Constraints [8]

The Hamiltonian is $H_{2,\eta\epsilon} = \int d^3x\, \mathcal{H}_{2,\eta\epsilon}$ with [13, 14]

$$\begin{aligned} \mathcal{H}_{2,\eta\epsilon} &= \pi_2^{nm}\pi_{2,nm} - \frac{1}{2}\pi_{2n}^{\ n}\pi_{2m}^{\ m} + \frac{1}{2}\pi_{2n}^{\ n}\partial^m h_{m0} - \frac{1}{2}\partial^k h^{n0}\partial_k h_{n0} \\ &-\frac{1}{4}\partial^k h^{nm}\partial_k h_{nm} + \frac{1}{8}\partial_n h^{n0}\partial^m h_{m0} + \frac{1}{2}\partial_n h^{nm}\partial^k h_{km} \\ &+\frac{1}{2}\partial_m h^{00}\partial^m h^n_n + \frac{1}{4}\partial_m h^n_n\partial^m h^k_k - \frac{1}{2}\partial_n h^{nm}\partial_m h_{00} - \frac{1}{2}\partial_n h^{nm}\partial_m h^k_k \\ &+\frac{1}{2}M_2^2 h^{n0}h_{n0} + \frac{1}{4}M_2^2 h^{nm}h_{nm} - \frac{1}{2}M_2^2 h^0_0 h^m_m - \frac{1}{4}M_2^2 h^n_n h^m_m \\ &-\frac{1}{2}cM_2^2\eta^\mu\eta_\mu - M_2\partial_n h^{n0} - M_2\partial_n h^{nm}\eta_m - M_2^2 h^0_0\epsilon - M_2^2 h^k_k\epsilon \\ &+\lambda_{2,00}\theta_2^{00} + \lambda_{2,0m}\theta_2^{0m} + \lambda_{0,\eta}\theta^0_\eta + \lambda_{m,\eta}\theta^m_\eta + \lambda_\epsilon\theta_\epsilon\,. \end{aligned} \tag{3.7}$$

In Ref. [13, 14] the primary, secondary, and tertiar constraints are worked out giving the Lagrange parameters λ in (3.7). Furthermore, here also the Dirac-brackets (DB's) are derived, enabling to establish the quantization.

[8] The material in this section is taken from Ref.'s [13, 14] and is included here for completeness.

The ETC relations are obtained by considering the fields as operators, replacing the Db's by commutators, and adding a factor i. The result is

$$
\begin{aligned}
\left[h^{00}(x), h^{0l}(y)\right]_0 &= \frac{4i}{3M_2^4}\partial^j\partial_j\partial^l\delta^3(x-y),\\
\left[h^{0m}(x), h^{kl}(y)\right]_0 &= \frac{-i}{M_2^2}\left[\frac{4}{3M_2^2}\partial^m\partial^k\partial^l - \frac{2}{3}\partial^m g^{kl} + \partial^k g^{ml} + \partial^l g^{mk}\right]\delta^3(x-y),\\
\left[\dot{h}^{00}(x), h^{00}(y)\right]_0 &= -\frac{4i}{3M_2^4}\partial^i\partial_i\partial^j\partial_j\delta^3(x-y),\\
\left[\dot{h}^{0m}(x), h^{0l}(y)\right]_0 &= \frac{i}{M_2^2}\left[\frac{4}{3M_2^2}\partial^m\partial^l\partial^j\partial_j + \frac{1}{3}\partial^m\partial^l + \partial^j\partial_j g^{ml}\right]\delta^3(x-y),\\
\left[\dot{h}^{00}(x), h^{kl}(y)\right]_0 &= \frac{i}{M_2^2}\left[\frac{4}{3M_2^2}\partial^k\partial^l\partial^j\partial_j + 2\partial^k\partial^l - \frac{2}{3}\partial^j\partial_j g^{kl}\right]\delta^3(x-y),\\
\left[\dot{h}^{nm}(x), h^{kl}(y)\right]_0 &= i\Big[-g^{nk}g^{ml} - g^{nl}g^{mk} + \frac{2}{3}g^{nm}g^{kl}\\
&\quad -\frac{1}{M_2^2}\left(\partial^n\partial^k g^{ml} + \partial^m\partial^k g^{nl} + \partial^n\partial^l g^{mk} + \partial^m\partial^l g^{mk}\right)\\
&\quad +\frac{2}{3M_2^2}\left(\partial^n\partial^m g^{kl} + g^{nm}\partial^k\partial^l\right) - \frac{4}{3M_2^4}\partial^n\partial^m\partial^k\partial^l\Big]\delta^3(x-y). \qquad (3.8)
\end{aligned}
$$

The ETC containing the auxiliary fields $\eta^\mu(x)$ and $\epsilon(x)$ are

$$
\begin{aligned}
\left[h^{00}(x), \eta^0(y)\right]_0 &= \frac{3i}{M_2(3+b)}\,\delta^3(x-y),\\
\left[h^{0n}(x), \eta^m(y)\right]_0 &= \frac{i}{M_2}\,g^{nm}\,\delta^3(x-y),\\
\left[h^{0n}(x), \epsilon(y)\right]_0 &= -\frac{i}{M_2^2}\left(\frac{1-b}{3+b}\right)\partial^n\delta^3(x-y),\\
\left[h^{mn}(x), \eta^0(y)\right]_0 &= -\frac{i}{M_2(3+b)}\,g^{nm}\,\delta^3(x-y),\\
\left[\eta^0(x), \eta^m(y)\right]_0 &= \frac{i}{M_2(3+b)}\,\partial^m\delta^3(x-y),\\
\left[\eta^0(x), \epsilon(y)\right]_0 &= \frac{3i}{2M_2}\,\frac{(1-b)}{(3+b)^2}\delta^3(x-y). \qquad (3.9)
\end{aligned}
$$

Here, as mentioned in [13], not shown are the ETC's among time derivatives of the fields in (3.9), which are of importance for the calculation of the non-equal times commutation (NETC) relations below. These follow from the constraints $\partial_\mu h^{\mu\nu}(x) = -bM_2\eta^\nu(x)$, $\partial\cdot\eta(x) = 4M_2(1-b)^{-1}\epsilon(x)$ in (2.10) for A=-1, which solves the time derivatives

$$
\begin{aligned}
\dot{h}^{00} &= \partial_n h^{0n} - bM_2\eta^0\,,\ \dot{h}^{0n} = \partial_k h^{kn} - bM_2\eta^0,\\
\dot{\eta}^0 &= \partial_k\eta^k + 4M_2(1-b)^{-1}\,\epsilon. \qquad (3.10)
\end{aligned}
$$

This enables the derivation of the ETC's (and NETC's) for these time derivatives in terms of those in (3.8) and (3.9) immediately.

IV. FIELD-COMMUTATORS AND SPIN-2 FIELD PROPAGATOR

The solutions of the homogeneous equations in (2.17) and (2.18) satisfy the identities, see [8, 13],

$$\begin{aligned}
h^{\mu\nu}(x) \;=\; & \int d^3z \left[\partial_0^z \Delta(x-z;M_2^2)\cdot h^{\mu\nu}(z) - \Delta(x-z;M_2^2)\,\partial_0^z h^{\mu\nu}(z)\right] + \frac{1}{M_2^2-M_\rho^2}\cdot \\
& \times \int d^3z \Big[\partial_0^z\Big(\Delta(x-z;M_\rho^2)-\Delta(x-z;M_2^2)\Big) - \Big(\Delta(x-z;M_\rho^2)-\Delta(x-z;M_2^2)\Big)\partial_0^z\Big]\cdot \\
& \times(\Box+M_2^2)\,h^{\mu\nu}(z) + \frac{1}{(M_\rho^2-M_\epsilon^2)(M_2^2-M_\rho^2)(M_2^2-M_\epsilon^2)}\cdot \\
& \times \int d^3z \Big[\partial_0^z\Big((M_2^2-M_\rho^2)\Delta(x-z;M_\epsilon^2)-(M_2^2-M_\epsilon^2)\Delta(x-z;M_\rho^2)+(M_\rho^2-M_\epsilon^2)\Delta(x-z;M_2^2)\Big) \\
& -\Big((M_2^2-M_\rho^2)\Delta(x-z;M_\epsilon^2)-(M_2^2-M_\epsilon^2)\Delta(x-z;M_\rho^2)+(M_\rho^2-M_\epsilon^2))\Delta(x-z;M_2^2)\Big)\partial_0^z\Big]\cdot \\
& \times(\Box+M_\rho^2)(\Box+M_2^2)\,h^{\mu\nu}(z).
\end{aligned} \tag{4.1}$$

and

$$\rho^\mu(x) \;=\; \int d^3z \left[\partial_0^z\Delta(x-z;M_\rho^2)\,\rho^\mu(z) - \Delta(x-z;M_\rho^2)\,\partial_0^z\rho^\mu(z)\right] \tag{4.2}$$

$$\epsilon(x) \;=\; \int d^3z \left[\partial_0^z\Delta(x-z;M_\epsilon^2)\,\epsilon(z) - \Delta(x-z;M_\epsilon^2)\,\partial_0^z\epsilon(z)\right] \tag{4.3}$$

Using this identity the commutation relation can be calculated from the ETC in Eqn's (3.8-(3.10) with the result

$$\begin{aligned}
\left[h^{\mu\nu}(x),h^{\alpha\beta}(y)\right] \;=\; & \Big\{\Big(\eta^{\mu\alpha}\eta^{\nu\beta}+\eta^{\mu\beta}\eta^{\nu\alpha}-\frac{2}{3}\eta^{\mu\nu}\eta^{\alpha\beta}\Big) \\
& +\frac{1}{M_2^2}\left(\partial^\mu\partial^\alpha\eta^{\nu\beta}+\partial^\nu\partial^\alpha\eta^{\mu\beta}+\partial^\mu\partial^\beta\eta^{\nu\alpha}+\partial^\nu\partial^\beta\eta^{\mu\alpha}\right) \\
& -\frac{2}{3M_2^2}\left(\partial^\mu\partial^\nu\eta^{\alpha\beta}+\eta^{\mu\nu}\partial^\alpha\partial^\beta\right)+\frac{4}{3M_2^2}\partial^\mu\partial^\nu\partial^\alpha\partial^\beta\Big\}\, i\Delta(x-y;M_2^2) \\
& -\frac{1}{M_2^2}\Big\{\left(\partial^\mu\partial^\alpha\eta^{\nu\beta}+\partial^\nu\partial^\alpha\eta^{\mu\beta}+\partial^\mu\partial^\beta\eta^{\nu\alpha}+\partial^\nu\partial^\beta\eta^{\mu\alpha}\right) \\
& +\frac{4}{M_\rho^4}\partial^\mu\partial^\nu\partial^\alpha\partial^\beta\Big\}\, i\Delta(x-y;M_\rho^2) - \Big\{\frac{1}{3}\frac{b}{3+b}\eta^{\mu\nu}\eta^{\alpha\beta} \\
& -\frac{2}{3M_2^2}\left(\partial^\mu\partial^\nu\eta^{\alpha\beta}+\eta^{\mu\nu}\partial^\alpha\partial^\beta\right)+\frac{4(3+b)}{3bM_2^4}\partial^\mu\partial^\nu\partial^\alpha\partial^\beta\Big\}\, i\Delta(x-y;M_\epsilon^2) \\
\;=\; & 2P_2^{\mu\nu\alpha\beta}(\partial)\, i\Delta(x-y;M_2^2)+P_\rho^{\mu\nu\alpha\beta}(\partial)\, i\Delta(x-y;M_\rho^2) \\
& +P_\epsilon^{\mu\nu\alpha\beta}(\partial)\, i\Delta(x-y;M_\epsilon^2),
\end{aligned} \tag{4.4}$$

where $P_2(\partial)$ etc are (on mass-shell) spin projection operators, see *e.g.* [19].
Defining $\bar{\eta}^{\mu\nu}=\eta^{\mu\nu}+\partial^\mu\partial^\nu/M_2^2$, $\hat{\eta}^{\mu\nu}=\eta^{\mu\nu}+\partial^\mu\partial^\nu/M_\rho^2$, and $\widetilde{\eta}^{\mu\nu}=\eta^{\mu\nu}+\partial^\mu\partial^\nu/M_\epsilon^2$ the commutator

takes on the compact form

$$
\begin{aligned}
\left[h^{\mu\nu}(x), h^{\alpha\beta}(y)\right] &= \left[\bar{\eta}^{\mu\alpha}\bar{\eta}^{\nu\beta} + \bar{\eta}^{\mu\beta}\bar{\eta}^{\nu\alpha} - \frac{2}{3}\bar{\eta}^{\mu\nu}\bar{\eta}^{\alpha\beta}\right] i\Delta(x-y;M_2^2) \\
&\quad + \frac{b}{M_\rho^2}\left[\partial^\mu\partial^\alpha\hat{\eta}^{\nu\beta} + \partial^\nu\partial^\alpha\hat{\eta}^{\mu\beta} + \partial^\mu\partial^\beta\hat{\eta}^{\nu\alpha} + \partial^\nu\partial^\beta\hat{\eta}^{\mu\alpha}\right] i\Delta(x-y;M_\rho^2) \\
&\quad - \frac{b}{3+b}\left\{\frac{1}{3}\tilde{\eta}^{\mu\nu}\tilde{\eta}^{\alpha\beta} + \frac{1}{M_\epsilon^2}\left(\partial^\mu\partial^\nu\tilde{\eta}^{\alpha\beta} + \tilde{\eta}^{\mu\nu}\partial^\alpha\partial^\beta\right) + \frac{3}{M_\epsilon^4}\partial^\mu\partial^\nu\partial^\alpha\partial^\beta\right\} i\Delta(x-y;M_\epsilon^2) \\
&= 2P_2^{\mu\nu\alpha\beta}(\partial, M_2)\, i\Delta(x-y;M_2^2) + P_\rho^{\mu\nu\alpha\beta}(\partial, M_\rho)\, i\Delta(x-y;M_\rho^2) \\
&\quad + P_\epsilon^{\mu\nu\alpha\beta}(\partial, M_\epsilon)\, i\Delta(x-y;M_\epsilon^2),
\end{aligned} \tag{4.5}
$$

and the Feynman spin-2 propagator is

$$
\begin{aligned}
D_F^{\mu\nu\alpha\beta}(x-y) &= -i\langle 0|T\left[h^{\mu\nu}(x)h^{\alpha\beta}(y)\right]|0\rangle \\
&= 2P_2^{\mu\nu\alpha\beta}(\partial)\,\Delta_F(x-y;M_2^2) + P_\rho^{\mu\nu\alpha\beta}(\partial)\,\Delta_F(x-y;M_\rho^2) \\
&\quad + P_\epsilon 2^{\mu\nu\alpha\beta}(\partial)\,\Delta_F(x-y;M_\epsilon^2)
\end{aligned} \tag{4.6}
$$

The masses M_η and M_ϵ are given in (2.13) and (2.14), see also([13]. In taking the limit $b \to \infty$ we use the form in Eqn. (4.5), which is important when studying the spectral representation lateron.
The spin-2 Feynman propagator, in the absence of the auxiliary fields, becomes [13]

$$
\begin{aligned}
D_F^{\mu\nu\alpha\beta}(x-y) &= -i\langle 0|T\left[h^{\mu\nu}(x)h^{\alpha\beta}(y)\right]|0\rangle \\
&= -i\theta(x^0-y^0)\, 2P_2^{\mu\nu\alpha\beta}(\partial)\Delta^{(+)}(x-y;M_2^2) \\
&\quad -i\theta(y^0-x^0)\, 2P_2^{\mu\nu\alpha\beta}(\partial)\Delta^{(-)}(x-y;M_2^2) \\
&= 2P_2^{\mu\nu\alpha\beta}(\partial)\,\Delta_F(x-y;M_2^2) + (\mathrm{non-covariant\ local\ terms})
\end{aligned} \tag{4.7}
$$

In the following, we often denote the mass by $M_2 \equiv M$. For the normalization of our solutions, the commutation relations of the field operators are important. Using the Dirac quantization method, and using a vector and a scalar auxiliary field, the obtained field commutators read

Refs. [13, 14] [9]

$$
\begin{aligned}
[\epsilon(x),\epsilon(y)] &= -\frac{3}{4}\frac{b(1-b)^2}{(3+b)^3}\left(\frac{M^2}{\mathcal{M}^2}\right)^2 i\Delta(x-y;M_\epsilon^2), && (4.8a)\\
[\eta^\mu(x),\epsilon(y)] &= -\frac{3}{2}\frac{(1-b)}{(3+b)^2}\frac{M^2}{\mathcal{M}^2}\frac{\partial^\mu}{\mathcal{M}}\, i\Delta(x-y;M_\epsilon^2), && (4.8b)\\
[\eta^\mu(x),\eta^\nu(y)] &= \left[\eta^{\mu\nu}+\frac{\partial^\mu\partial^\nu}{M_\eta^2}\right] i\Delta(x-y;M_\eta^2) \\
&\quad +\frac{3}{b(3+b)}\frac{\partial^\mu\partial^\nu}{\mathcal{M}^2}\, i\Delta(x-y;M_\epsilon^2), && (4.8c)\\
[\epsilon(x),h^{\mu\nu}(y)] &= \frac{(1-b)}{(3+b)}\left[\frac{\partial^\mu\partial^\nu}{M^2}-\frac{1}{2}\frac{b}{(3+b)}\eta^{\mu\nu}\right] i\Delta(x-y;M_\epsilon^2), && (4.8d)\\
[\eta^\alpha(x),h^{\mu\nu}(y)] &= \frac{1}{M}\left[\partial^\mu\eta^{\alpha\nu}+\partial^\nu\eta^{\alpha\mu}+\frac{2}{M_\eta^2}\partial^\alpha\partial^\mu\partial^\nu\right] i\Delta(x-y;M_\eta^2) \\
&\quad -\frac{1}{M}\left[\frac{1}{(3+b)}\partial^\alpha\eta^{\mu\nu}+\frac{2}{M_\eta^2}\partial^\alpha\partial^\mu\partial^\nu\right] i\Delta(x-y;M_\epsilon^2), && (4.8e)\\
\left[h^{\mu\nu}(x),h^{\alpha\beta}(y)\right] &= \left[\left(\eta^{\mu\alpha}\eta^{\nu\beta}+\eta^{\mu\beta}\eta^{\nu\alpha}\right)-\frac{2}{3}\eta^{\mu\nu}\eta^{\alpha\beta}+\ldots\right] i\Delta(x-y;M^2) \\
&\quad -\left[\frac{1}{3}\frac{b}{3+b}\eta^{\mu\nu}\eta^{\alpha\beta}+\ldots\right] i\Delta(x-y;M_\epsilon^2). && (4.8f)
\end{aligned}
$$

The ellipsis in the square brackets above denote terms with $\partial^\mu, ..., \partial^\beta$. These are taking care off in section VII and are unimportant due to $\partial_\mu t_M^{\mu\nu}(x)=0$ and $(t_M)_\mu^\mu(x)=0$.
The field commutators for the $h^{\mu\nu}(x)$, $\rho^\mu(x)$, and $\epsilon(x)$ fields are readily derived. Starting from Eqs. (4.1) one obtains

$$
\begin{aligned}
[\epsilon(x),\epsilon(y)] &= -\frac{3}{4}\frac{b(1-b)^2}{(3+b)^3}\left(\frac{M^2}{\mathcal{M}^2}\right)^2 i\Delta(x-y;M_\epsilon^2), && (4.9a)\\
[\rho^\mu(x),\epsilon(y)] &= 0, && (4.9b)\\
[\rho^\mu(x),\rho^\nu(y)] &= \left[\eta^{\mu\nu}-\frac{\partial^\mu\partial^\nu}{bM^2}\right] i\Delta(x-y;M_\rho^2), && (4.9c)\\
[\rho^\alpha(x),h^{\mu\nu}(y)] &= \frac{1}{M}\left[(\partial^\mu\eta^{\alpha\nu}+\partial^\nu\eta^{\alpha\mu})-\frac{2}{bM^2}\partial^\alpha\partial^\mu\partial^\nu\right] i\Delta(x-y;M_\rho^2), && (4.9d)\\
[\epsilon(x),h^{\mu\nu}(y)] &= \frac{(1-b)}{(3+b)}\left[\frac{\partial^\mu\partial^\nu}{M^2}-\frac{1}{2}\frac{b}{(3+b)}\eta^{\mu\nu}\right] i\Delta(x-y;M_\epsilon^2), && (4.9e)\\
\left[h^{\mu\nu}(x),h^{\alpha\beta}(y)\right] &= \left[\left(\eta^{\mu\alpha}\eta^{\nu\beta}+\eta^{\mu\beta}\eta^{\nu\alpha}\right)-\frac{2}{3}\eta^{\mu\nu}\eta^{\alpha\beta}+\ldots\right] i\Delta(x-y;M^2), \\
&\quad -\left[\frac{1}{3}\frac{b}{3+b}\eta^{\mu\nu}\eta^{\alpha\beta}+\ldots\right] i\Delta(x-y;M_\epsilon^2), && (4.9f)
\end{aligned}
$$

[9] In contrast to [14] we use here for the Minkowski-metric the notation $\eta^{\mu\nu}$. There should be no confusion with the auxiliary vector-field $\eta^\mu(x)$.

where $M_\rho = M_\eta$, and in the commutator of ρ^α and $h^{\mu\nu}$ we have set $\mathcal{M} = M$, anticipating with what will be done later. Again, the ellipsis in the square brackets above denote terms with $\partial^\mu, ..., \partial^\beta$, which do not contribute because of $\partial_\mu t_M^{\mu\nu}(x) = 0$ and $(t_M)^\mu_\mu(x) = 0$. [10]
It is important to note that upon quantization, the sign in $[\rho^\mu(x), \rho^\nu(y)]$ on the r.h.s. means negative norm. Likewise for $b/(3+b) > 0$ we have negative norm states for $\epsilon(x)$. This would require to set up for physical states $|f\rangle$ the *subsidiary conditions* for the positive-frequency parts

$$\rho^{\mu(+)}(x)|f\rangle = 0 \ , \ \ \epsilon^{(+)}(x)|f\rangle = 0. \tag{4.10}$$

It is one of the aims of this investigation to find a theory which allows (i) a smooth and correct massless limit, and (ii) a perturbation expansion in the small mass M.
For the latter to be meaningful, it is necessary that the theory satisfies the following requirements: (i) no-ghosts, (ii) unitarity, and (iii) a correct massless limit. This would open the possibility of giving a small mass to the graviton without destroying e.g. the correct prediction for the perihelium of Mercury.

Corollary: As follows from the commutator (4.9f) the correct masless limit requires the double limit $b \to \pm\infty, M \to 0$.

V. MASSLESS LIMIT: SCALAR-TENSOR OR IMAGINARY-GHOST THEORY

In the previous section we concluded that the double limit: $M \to 0; b \to \infty$ leads to the proper massless graviton propagator. It is the purpose of this section to analyze the distinctive physical contents of the propagator for the $h^{\mu\nu}$-field for the different regions of the parameter $-\infty < b < +\infty$. In particularly in the massless limit we want to investigate the possibility of a smooth decoupling of the "false helicities".
From the commutators in (4.9) we obtain the propagator by the replacement

$$\Delta(x-y; M^2) \to \Delta_F(x-y; M^2)\,. \tag{5.1}$$

Then, we have apart from irrelevant terms for the $h^{\mu\nu}$-field the Feynman propagator [11]

$$\begin{aligned} D_F^{\mu\nu,\alpha\beta}(x-y) \;=\; & \left[\frac{1}{2}\left(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha}\right) - \frac{1}{3}\eta^{\mu\nu}\eta^{\alpha\beta}\right] \Delta_F(x-y; M^2) \\ & -\frac{1}{6}\frac{b}{3+b}\eta^{\mu\nu}\eta^{\alpha\beta}\, \Delta_F(x-y; -\frac{2b}{3+b}M^2)\,. \end{aligned} \tag{5.2}$$

Introducing the parameter λ, defined as

$$\lambda =: \ b/(3+b), \tag{5.3}$$

we can distinghuish three regions, see Fig. 1, for the b-parameter

$$\begin{aligned} I \ &: \quad 0 \ \leq b < +\infty \ (\ 0 \leq \lambda \leq 1), \\ II \ &: \ -3 \ \leq b \leq 0 \quad \ (-\infty < \lambda \leq 0), \\ III \ &: \ -\infty < b \leq -3 \ \ (1 < \lambda < +\infty). \end{aligned}$$

---

[10] Later on we will consider the double limit $b \to \infty$ and $M \to 0$. In doing so we keep $M_\rho^2 = M_\eta^2 = -bM^2$ finite.
[11] In the limit $M \to 0, b \to \infty$ this propagator becomes the standard one, see e.g. [20].

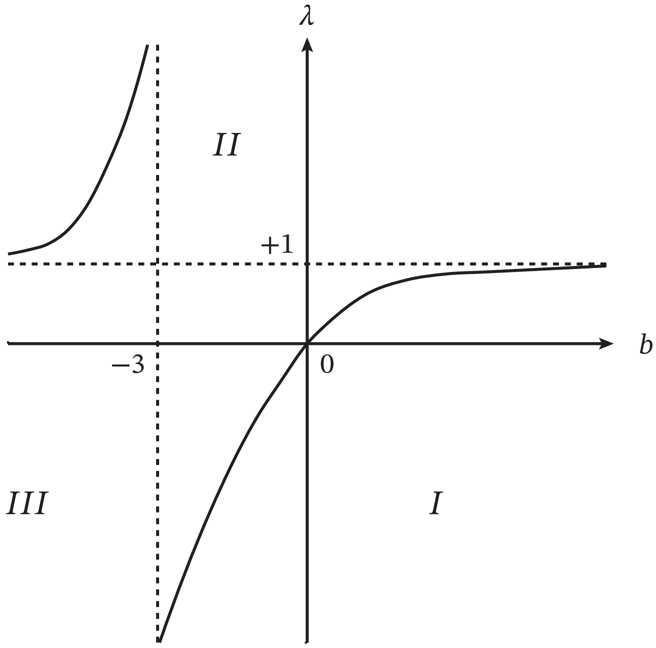

FIG. 1: Three regions in (b,λ)-space. Region I, III: spin-2 and spin-0 imaginary ghost. Region II: (massive) spin-2 and spin-0.

One sees from (5.2) that for $M \neq 0$ for region II the contents is a massive spin-2, and a massive spin-0 particle. For regions I and III the contents is besides a massive spin-2, a spin-0 ghost particle with an imaginary mass.
The main goal of this investigation to find a theory which allows (i) a smooth massless limit, and (ii) a perturbation expansion in the small mass M. In a no-ghost scenario it is necessary that the theory satisfies the following requirements:
1. No-ghost: $M_\varepsilon^2 > 0 \to b/(3+b) < 0$,
2. Unitarity: $b/(3+b) < 0$,
3. Correct massless limit: $b/(3+b) \to +1 - \Delta,\ \ \Delta > 0$,
where $\Delta \equiv 3/(3+b)$.
Clearly, requirement 3) is in conflict with 1) and 2) if $\Delta = 0$, i.e. for a pure spin-2 theory in the massless limit. So, with exclusively physical fields at best we can end up with is a scalar-tensor type theory!

In order to analyze the massless limit in more detail we write the $h^{\mu\nu}$-propagator as follows

$$
\begin{aligned}
D_F^{\mu\nu,\alpha\beta}(x-y) &= \left[\frac{1}{2}\left(\eta^{\mu\alpha}\eta^{\nu\beta}+\eta^{\mu\beta}\eta^{\nu\alpha}\right)-\frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\right]\Delta_F(x-y;M^2)+ \\
&\quad \frac{1}{6}\eta^{\mu\nu}\eta^{\alpha\beta}\left[\Delta_F(x-y;M^2)-\lambda\,\Delta_F(x-y;-2\lambda\,M^2)\right] \\
&\equiv \bar{D}_F^{\mu\nu,\alpha\beta}(x-y,M^2)+\Delta D_F^{\mu\nu,\alpha\beta}(x-y;\lambda M^2) \qquad (5.4)
\end{aligned}
$$

In the limit $M \to 0$ for $\lambda = 1$ ($b \to \pm\infty$) the extra piece $\Delta D_F^{\mu\nu;\alpha\beta} \to 0$, and we get the proper massless spin-2 propagator. Then, with $\lambda = 1$ we have for $M \neq 0$ a theory with (i) a massive spin-2, and (ii) an "imaginary" spin-0 ghost particle. Below we will show that the latter will satisfy a free field equation, which can be quantized [6] and taken care off using a *Gupta-type subsidiary condition*, see below in section A.

In momentum space we have

$$\begin{aligned}\Delta\widetilde{F}_F^{\mu\nu;\alpha\beta}(p) &= \frac{i}{6}\eta^{\mu\nu}\eta^{\alpha\beta}\left[\frac{1}{p^2-M^2+i\delta}-\frac{\lambda}{p^2+2\lambda M^2+i\delta}\right] \\ &= \frac{1}{6}(1-\lambda)\eta^{\mu\nu}\eta^{\alpha\beta}\frac{p^2+3\lambda M^2/(1-\lambda)}{p^2+2\lambda M^2+i\delta}\cdot\frac{1}{p^2-M^2+i\delta}\,. \end{aligned} \tag{5.5}$$

Preliminary summary and prospect: We found in this section that by choosing the constants suitably, and performing a couple of gauge transformations, we can eliminate the unwanted helicity components in the massless limit. Thereby we arrive at a satisfactorily massless spin-2 theory. This in accordance with the Dirac quantization method for spin-2 fields using auxiliary vector and scalar (ghost) fields.
We found theories of the kind: (i) $-\infty < \lambda \leq 0$: massive spin-2 and spin-0 particles with in the massless limit a kind of scalar-tensor" model, (ii) $\lambda = 1$: a massive spin-2 and an imaginary-ghost spin-0 particles with a proper massless limit giving a relativistic theory of gravitation in Minkowski space (LRGT-AF). The latter we will investigate further in this paper to find out whether it is posible to give a small mass to the graviton, without destroying the correct prediction for the perihelium of Mercury. This in contrast to the formalism considered by Van Dam and Veltman [1, 2].

VI. ELIMINATION $\rho(x)$-FIELD, NEW FIELD COMMUTATORS

The the conditions for the massless limit leads to the limit $b \to \pm\infty$ as the only solution, see Eqn. (4.9f). For the proper behavior of the propagators the ρ-field is not necessary, and therefore we effectively eliminate the ρ-field from the equations by restricting the Hilbert-space (I). Also, this can be done by out-integrating the ρ-field (II). This leaves finally in the spin-2 model of this paper, besides the $h^{\mu\nu}$-fields only the scalar-ghost field $\epsilon(x)$ with the subsidiary Gupta condition $\epsilon^{(+)}(x)|\Psi\rangle = 0$ for the states $|\Psi\rangle \in \mathcal{V}(h,\epsilon)$.

A. Elimination $\rho(x)$-field

Elimination $\rho(x)$-field I: The total Hilbert space $\mathcal{V}_T = \mathcal{V}(h,\rho,\epsilon)$ is an indefinite-metric space. From now on we restrict ourselves to $\mathcal{V}(h,\epsilon) \subset \mathcal{V}_T$, *i.e.* states $|\Psi\rangle \in \mathcal{V}(h,\epsilon)$ which do not contain ρ-field quanta. Then, for all (field) equations we consider states such that $\langle\Phi|f(\rho)|\Psi\rangle = 0$, $\forall f$ with $f(0) = 0$. Therefore, in the rest of this work $\rho(x) \equiv 0$.
Elimination $\rho(x)$-field II: Notice that the $\rho(x)$-field occurs only in $\mathcal{L}_{GF}$ in a quadratic form without derivatives. Consider the generating functional

$$Z = \int \mathcal{D}h_{\mu\nu}\mathcal{D}\rho_\lambda\mathcal{D}\epsilon\ \exp\left[(i/\hbar)\int \mathcal{L}\ d^4x\right],$$

where $\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_{GF}$. Ignoring 1/b-terms the gauge-fixing Lagrangian is

$$\mathcal{L}_{GF} = \frac{1}{2} b\, \mathcal{M}^2 \rho^\mu(x)\rho_\mu(x) + \mathcal{M}\left[\partial_\mu h^{\mu\nu}(x) + \widetilde{\gamma}\mathcal{M}\,\partial^\nu \epsilon(x)\right] \cdot \rho_\nu(x) + \mathcal{M}^2 h(x)\epsilon(x).$$

Introducing $\chi^\nu(x) = \partial_\mu h^{\mu\nu}(x) + \widetilde{\gamma}\mathcal{M}\,\partial^\nu\epsilon(x)$, the $\rho(x)$-field can be out-integrated and yields

$$\begin{aligned}
&\int \mathcal{D}\rho_\lambda\ \exp\left\{(i/\hbar)\int d^4x \left[\frac{1}{2} b\mathcal{M}^2 \rho^\mu(x)\rho_\mu(x) + \mathcal{M}\ \chi^\nu(x)\rho_\nu(x)\right]\right\} \\
&= \mathcal{N}\exp\left\{(i/\hbar)\int d^4x\ \left[-\frac{b\mathcal{M}^2}{2b^2\mathcal{M}^2}\chi^\mu(x)\chi_\mu(x)\right]\right\}
\end{aligned}$$

This means that $\mathcal{L}_{GF} \to \mathcal{L}'_{GF}$ where

$$\mathcal{L}'_{GF} \ = \ -\frac{b\mathcal{M}^2}{2b^2\mathcal{M}^2}\left(\partial_\mu h^{\mu\nu} + \widetilde{\gamma}\mathcal{M}\partial^\nu\epsilon\right)\left(\partial^\alpha h_{\alpha\nu} + \widetilde{\gamma}\mathcal{M}\partial_\nu\epsilon\right) + \mathcal{M}^2 h(x)\,\epsilon(x).$$

So, for $b \to \infty$ the first (contact) interaction vanishes, only the second one survives. This defines the model for $b \to \infty$ without the ρ-field. (QED)

Remark: $\mathcal{N} \sim 1/\sqrt{b} \to 0$. So the functional integral vanishes for $b = \infty$. This is consistent with the Riemann-Lebesque lemma. Therefore, in the limit $\rho(x)$ has to be set zero.

B. Commutators and Propagator spin-2 for $b \to \infty$

Since we are interested in the correct massless limit we take $b \to \pm\infty$, i.e. $\lambda = 1$. Leaving out the ρ-field terms the commutators in Eqn. (4.4) become

$$[\epsilon(x), \epsilon(y)] \ = \ -\frac{3}{4}\left(\frac{M^2}{\mathcal{M}^2}\right)^2\ i\Delta(x-y; M_\epsilon^2), \tag{6.1a}$$

$$[\epsilon(x), h^{\mu\nu}(y)] \ = \ -\left[\frac{\partial^\mu\partial^\nu}{M^2} - \frac{1}{2}\eta^{\mu\nu}\right]\ i\Delta(x-y; M_\epsilon^2), \tag{6.1b}$$

$$\begin{aligned}
\left[h^{\mu\nu}(x), h^{\alpha\beta}(y)\right] \ = \ & \left\{\left(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \frac{2}{3}\eta^{\mu\nu}\eta^{\alpha\beta}\right)\right. \\
& + \frac{1}{M_2^2}\left(\partial^\mu\partial^\alpha\eta^{\nu\beta} + \partial^\nu\partial^\alpha\eta^{\mu\beta} + \partial^\mu\partial^\beta\eta^{\nu\alpha} + \partial^\nu\partial^\beta\eta^{\mu\alpha}\right) \\
& \left. - \frac{2}{3M_2^2}\left(\partial^\mu\partial^\nu\eta^{\alpha\beta} + \eta^{\mu\nu}\partial^\alpha\partial^\beta\right) + \frac{4}{3M_2^4}\partial^\mu\partial^\nu\partial^\alpha\partial^\beta\right\} i\Delta(x-y; M_2^2) \\
& - \left\{\frac{1}{3}\eta^{\mu\nu}\eta^{\alpha\beta} - \frac{2}{3M_2^2}\left(\partial^\mu\partial^\nu\eta^{\alpha\beta} + \eta^{\mu\nu}\partial^\alpha\partial^\beta\right) + \frac{4}{3M_2^4}\partial^\mu\partial^\nu\partial^\alpha\partial^\beta\right\} i\Delta(x-y; M_\epsilon^2).
\end{aligned} \tag{6.1c}$$

The compact form of (6.1c) reads

$$\begin{aligned}
\left[h^{\mu\nu}(x), h^{\alpha\beta}(y)\right] \ = \ & \left\{\bar{\eta}^{\mu\alpha}\bar{\eta}^{\nu\beta} + \bar{\eta}^{\mu\beta}\bar{\eta}^{\nu\alpha} - \frac{2}{3}\bar{\eta}^{\mu\nu}\bar{\eta}^{\alpha\beta}\right\} i\Delta(x-y; M_2^2) \\
& - \left\{\frac{1}{3}\widetilde{\eta}^{\mu\nu}\widetilde{\eta}^{\alpha\beta} + \frac{1}{M_\epsilon^2}\left(\partial^\mu\partial^\nu\widetilde{\eta}^{\alpha\beta} + \widetilde{\eta}^{\mu\nu}\partial^\alpha\partial^\beta\right) + \frac{3}{M_\epsilon^4}\partial^\mu\partial^\nu\partial^\alpha\partial^\beta\right\} i\Delta(x-y; M_\epsilon^2),
\end{aligned} \tag{6.2}$$

with the notation $\bar{\eta}^{\mu\nu} \equiv \eta^{\mu\nu} + \partial^\mu\partial^\nu/M^2$ and $\tilde{\eta}^{\mu\nu} \equiv \eta^{\mu\nu} + \partial^\mu\partial^\nu/M_\epsilon^2$. For the spin-2 Feynman propagator from (4.6) one obtains

$$\begin{aligned} D_F^{\mu\nu\alpha\beta}(x-y) &= -i\langle 0|T\left[h^{\mu\nu}(x)h^{\alpha\beta}(y)\right]|0\rangle = 2P_2^{\mu\nu\alpha\beta}(\partial)\,\Delta_F(x-y;M_2^2) \\ &\quad +P_\epsilon^{\mu\nu\alpha\beta}(\partial)\,\Delta_F(x-y;M_\epsilon^2) \end{aligned} \tag{6.3}$$

because $\lim_{b\to\infty}\Delta_F(x-y;M_\rho^2) = 0$.

The derivation of the sum rules for the spectral function $\rho(s)$, see below, uses for example the ETC relations

$$\left[h^{00}(x), h^{0l}(y)\right]_0 = +\frac{2i}{M_2^2}\partial^l\delta^3(\mathbf{x}-\mathbf{y}), \tag{6.4a}$$

$$\left[h^{0m}(x), h^{kl}(y)\right]_0 = -\frac{i}{M_2^2}\left(\eta^{ml}\partial^k + \eta^{mk}\partial^l\right)\delta^3(\mathbf{x}-\mathbf{y}) \tag{6.4b}$$

Here, the difference with [13] Eqn. (21) is due to the presence of the $\epsilon(x)$-field.

VII. ASYMPTOTIC FIELDS. PHYSICAL CONTENTS $h_{\mu\nu}$-FIELD

A. Massive $M_2 \neq 0$ case

Coupling the $h_{\mu\nu}$-field to the conserved matter and traceless energy-momentum tensor gives the field equations, setting $\mathcal{M} = M$ and $\alpha_2 \equiv -2\lambda \to -2$,

$$\left(\Box + M^2\right)\, h_{\mu\nu}(x) + 2M^2\left(\eta_{\mu\nu} + \alpha_2\frac{\partial_\mu\partial_\nu}{M^2}\right)\epsilon(x) = \kappa\, t_{M,\mu\nu}(x), \tag{7.1a}$$

$$\left(\Box + \alpha_2 M^2\right)\,\epsilon(x) = 0,\ \partial^\mu h_{\mu\nu}(x) + \alpha_2\partial_\nu\epsilon(x) = 0. \tag{7.1b}$$

Requiring $(\Box + M^2)\, h^\mu_\mu(x) = 0$ implies that $(t_M)^\mu_\mu = 0$. With the spectral representation

$$\kappa^2\left(0|\left[t_{M,\mu\nu}(x), t_{M,\alpha\beta}(y)\right]|0\right) = i\int_{0+}^\infty ds\; s^2\pi(s)\,\bar{\Pi}_{\mu\nu;\alpha\beta}(s)\,\Delta(x-y;s) \tag{7.2}$$

where

$$\bar{\Pi}_{\mu\nu;\alpha\beta}(s) = \left\{\left(\bar{\eta}_{\mu\alpha}\bar{\eta}_{\nu\beta} + \bar{\eta}_{\mu\beta}\bar{\eta}_{\nu\alpha}\right) - \frac{2}{3}\bar{\eta}_{\mu\nu}\bar{\eta}_{\alpha\beta}\right\} \tag{7.3}$$

with

$$\bar{\eta}_{\mu\nu}(\partial, s) = \eta_{\mu\nu} + s^{-1}\partial_\mu\partial_\nu,\ \partial^\mu\bar{\eta}_{\mu\nu} = 0,\ \eta^{\mu\nu}\bar{\eta}_{\mu\nu}(\partial) = 3. \tag{7.4}$$

Using $\left[t_{M,\mu\nu}(x), \epsilon(y)\right] = 0$ we get

$$\begin{aligned} \left(0|\left[h_{\mu\nu}(x), h_{\alpha\beta}(y)\right]|0\right) &= -\frac{i}{3}\left(\eta_{\mu\nu} + \alpha_2\frac{\partial_\mu\partial_\nu}{M^2}\right)\left(\eta_{\alpha\beta} + \alpha_2\frac{\partial_\alpha\partial_\beta}{M^2}\right)\Delta(x-y;\alpha_2 M^2) \\ &+ i\int_0^\infty ds\;\rho(s)\bar{\Pi}_{\mu\nu;\alpha\beta}(s)\,\Delta(x-y;s),\ \rho(s) = s(s-M^2)^{-2}\pi(s) \geq 0. \end{aligned} \tag{7.5}$$

Differentiation (7.5) by x^0 and setting $x^0 = y^0$ one obtains

$$\left(0\right|\left[\dot{h}_{\mu\nu}(x), h_{\alpha\beta}(y)\right]_0 \left|0\right) = -i\Big\{\bar{\Pi}_{\mu\nu;\alpha\beta}(M^2) - \frac{1}{3}\widetilde{\eta}_{\mu\nu}\widetilde{\eta}_{\alpha\beta}$$
$$-\frac{1}{M_\epsilon^2}\left(\partial_\mu\partial_\nu\widetilde{\eta}_{\alpha\beta} + \widetilde{\eta}_{\mu\nu}\partial_\alpha\partial_\beta\right) - \frac{3}{M_\epsilon^4}\partial_\mu\partial_\nu\partial_\alpha\partial_\beta\Big\}\,\delta^3(\mathbf{x}-\mathbf{y}) =$$
$$\Big[\frac{i}{3}\left(\eta_{\mu\nu} + \alpha_2\frac{\partial_\mu\partial_\nu}{M^2}\right)\left(\eta_{\alpha\beta} + \alpha_2\frac{\partial_\alpha\partial_\beta}{M^2}\right)\Delta(x-y;\alpha_2 M^2)$$
$$-i\int_0^\infty ds\,\rho(s)\bar{\Pi}_{\mu\nu;\alpha\beta}(s)\Big]\,\delta^3(\mathbf{x}-\mathbf{y}), \tag{7.6}$$

Taking $\mu,\nu,\alpha,\beta = m,n,k,l$ and comparing the $\eta_{mn}\eta_{kl}$ terms gives the sum rule

$$1 = \int_0^\infty ds\,\rho(s). \tag{7.7}$$

The ETC relations derived from (6.2) and (7.5) give the equations

$$\left(0\right|\left[h_{00}(x), h_{0l}(y)\right]_0 \left|0\right) = -\frac{i\alpha_2}{M^2}\partial_l\delta^3(\mathbf{x}-\mathbf{y}) =$$
$$\left[+\frac{i\alpha_2}{3M^2}\left((1-\alpha_2^2) + \alpha_2\boldsymbol{\nabla}^2/M^2\right) - \frac{4i}{3}\int_0^\infty ds\,(\rho(s)/s^2)\boldsymbol{\nabla}^2\right]\,\partial_l\delta^3(\mathbf{x}-\mathbf{y}),$$
$$\left(0\right|\left[h_{0m}(x), h_{kl}(y)\right]_0 \left|0\right) = -\frac{i}{M^2}\left(\eta_{ml}\partial_k + \eta_{mk}\partial_l\right)\delta^3(\mathbf{x}-\mathbf{y}) =$$
$$\left[+\frac{i\alpha_2}{3M^2}\left(\eta_{kl} + \frac{\alpha_2}{M^2}\partial_k\partial_l\right)\partial_m - i\int_0^\infty ds\,(\rho(s)/s)\Big\{\left(\eta_{ml}\partial_k + \eta_{mk}\partial_l - \frac{2}{3}\eta_{kl}\partial_m\right) + \frac{4}{3}s^{-1}\partial_k\partial_l\partial_m\Big\}\right]\,\delta^3(\mathbf{x}-\mathbf{y}).$$

With $\alpha_2 = -2$ these relations lead to the sum rules

$$(1/M^2) = \int_0^\infty ds\,\rho(s)/s\;,\;(1/M^4) = \int_0^\infty ds\,\rho(s)/s^2. \tag{7.8}$$

Let M_r be the physical ("renormalized") mass of $h_{\mu\nu}$, then

$$\rho(s) = Z\delta(s - M_r^2) + \theta(s - s_0)\sigma(s) \tag{7.9}$$

with $s_0 > 0$. The sum rules (7.7) and (7.8) give

$$0 < Z = 1 - \int_{s_0}^\infty ds\,\sigma(s) \le 1, \tag{7.10a}$$

$$\frac{1}{M^{2n}} = \frac{Z}{M_r^{2n}} + \int_{s_0}^\infty ds\,\sigma(s)/s^n\;(n = 1,2) \tag{7.10b}$$

which shows that if $M \to 0$ then $M_r \to 0$. Thus in this limit the $h_{\mu\nu}$-field can not acquire a non-zero physical mass. The condition for the validity is that $t_{M,\mu\nu}$ has no massless discrete spectrum when $M \neq 0$. This is the analoque of the same theorem as for the massive vector-field in the B-field formalism [8].

The commutation relations in (6.1) and before in this paper were for free fields, which we denote furtheron in this section as 'asymptotic" fields $h^{as}_{\mu\nu}(x), \epsilon^{as}(x)$, which satisfy (6.1) with $\mathcal{M} = M$, and (7.1).
The asymptotic field $h^{as}_{\mu\nu}$ and $\epsilon^a s$ are defined

$$Z^{-1/2} h_{\mu\nu} \to h^{as}_{\mu\nu}\ ,\ k_2\, Z^{1/2}\epsilon = \epsilon^{as}, \tag{7.11}$$

where

$$k_2 \ \equiv\ Z^{-1} M_r^2/M^2 = 1 + Z^{-1} M_r^2 \int_{s_0}^{\infty} ds\, \sigma(s)/s \geq 1, \tag{7.12a}$$

$$k_4 \ \equiv\ Z^{-1} M_r^4/M^4 = 1 + Z^{-1} M_r^4 \int_{s_0}^{\infty} ds\, \sigma(s)/s^2 \geq 1, \tag{7.12b}$$

where we also introduced k_4. Then, with $\rho(s) = Z\delta(s - M_r^2)$,

$$\begin{aligned}
\left[h^{as}_{\mu\nu}(x), h^{as}_{\alpha\beta}(y)\right] \ &=\ \frac{i}{3} Z^{-1}\left(\eta_{\mu\nu} + \alpha_2 \frac{\partial_\mu\partial_\nu}{M^2}\right)\left(\eta_{\alpha\beta} + \alpha_2 \frac{\partial_\alpha\partial_\beta}{M^2}\right)\Delta(x-y;\alpha_2 M^2) \\
&\quad + i\bar{\Pi}_{\mu\nu;\alpha\beta}(M_r^2)\ \Delta(x-y;M_r^2) \\
&=\ \frac{i}{3} Z^{-1}\left(\eta_{\mu\nu} + \bar{\alpha}_{2r} \frac{\partial_\mu\partial_\nu}{M_r^2}\right)\left(\eta_{\alpha\beta} + \bar{\alpha}_{2r} \frac{\partial_\alpha\partial_\beta}{M_r^2}\right)\Delta(x-y;\alpha_{2r} M_r^2) \\
&\quad + i\bar{\Pi}_{\mu\nu;\alpha\beta}(M_r^2)\ \Delta(x-y;M_r^2),
\end{aligned} \tag{7.13}$$

where $M^2\alpha_2 \ =\ M_r^2\alpha_{2r}$ and $\bar{\alpha}_{2r} \ \equiv\ (M_r/M)^4\alpha_{2r} \ =\ k_4 Z\ \alpha_{2r}$. Notice that the commutator is completely expressed as a function of "renormalized" quantities. For the other commutators we have

$$\left[\epsilon^{as}(x), h^{as}_{\alpha\beta}(y), \epsilon^{as}(y)\right] \ =\ +\frac{i}{2} k_2 \left(\eta_{\alpha\beta} + \bar{\alpha}_{2r} \frac{\partial_\alpha\partial_\beta}{M_r^2}\right)\Delta(x-y;\alpha_r M_r^2), \tag{7.14}$$

$$\left[\epsilon^{as}(x), \epsilon^{as}(y)\right] \ =\ -\frac{3i}{4}\ k_2 \left(\frac{M_r}{M}\right)^2 \Delta(x-y;\alpha_r M_r^2). \tag{7.15}$$

To determine the "renormalized" $h^{as}_{\mu\nu}$-equation we get from (7.13) the relation

$$\begin{aligned}
(\Box + M_r^2)\left[h^{as}_{\mu\nu}(x), h^{as}_{\alpha\beta}(y)\right] \ &=\ \frac{i}{3} Z^{-1}(1-\alpha_{2r}) M_r^2 \left(\eta_{\mu\nu} + \bar{\alpha}_{2r} \frac{\partial_\mu\partial_\nu}{M_r^2}\right)\left(\eta_{\alpha\beta} + \bar{\alpha}_{2r} \frac{\partial_\alpha\partial_\beta}{M_r^2}\right)\Delta(x-y;\alpha_{2r} M_r^2) \\
&=\ -\frac{2}{3}(k_2 Z)^{-1}(1-\alpha_{2r}) M_r^2 \left(\eta_{\mu\nu} + \bar{\alpha}_{2r} \frac{\partial_\mu\partial_\nu}{M_r^2}\right)\left[\epsilon^{as}(x), h^{as}_{\alpha\beta}(y)\right],
\end{aligned}$$

which leads to the "renormalized" equation

$$(\Box + M_r^2)\ h^{as}_{\mu\nu}(x) + \frac{2}{3}\,(1-\alpha_{2r})\,(k_2 Z)^{-1} M_r^2 \left(\eta_{\mu\nu} + \bar{\alpha}_{2r} \frac{\partial_\mu\partial_\nu}{M_r^2}\right)\ \epsilon^{as}(x) = 0. \tag{7.16}$$

Next we introduce the asymptotic physical spin-2 field by

$$\begin{aligned} U^{as}_{\mu\nu}(x) &= h^{as}_{\mu\nu}(x) + \frac{2}{3}(1-\alpha_{2r})(k_2 Z)^{-1} M_r^2 \left(\eta_{\mu\nu} + \bar{\alpha}_{2r}\frac{\partial_\mu\partial_\nu}{M_r^2}\right)(\Box + M_r^2)^{-1}\,\epsilon^{as}(x) \\ &= h^{as}_{\mu\nu}(x) + \frac{2}{3}(k_2 Z)^{-1}\left(\eta_{\mu\nu} + \bar{\alpha}_{2r}\frac{\partial_\mu\partial_\nu}{M_r^2}\right)\epsilon^{as}(x). \end{aligned} \tag{7.17}$$

which satisfies

$$\left[U^{as}_{\mu\nu}(x), U^{as}_{\alpha\beta}(y)\right] = \left\{(\bar{\eta}_{\mu\alpha}\bar{\eta}_{\nu\beta} + \bar{\eta}_{\mu\beta}\bar{\eta}_{\nu\alpha}) - \frac{2}{3}\bar{\eta}_{\mu\nu}\bar{\eta}_{\alpha\beta}\right\} i\Delta(x-y; M_r^2), \tag{7.18a}$$

$$\left[U^{as}_{\mu\nu}(x), \epsilon^{as}(y)\right] = 0, \tag{7.18b}$$

$$\left(\Box + M_r^2\right)\, U^{as}_{\mu\nu}(x) = 0,\ \partial^\mu U^{as}_{\mu\nu}(x) = 0, \tag{7.18c}$$

With the interaction the field equation reads $(\Box + M_r^2)U_{\mu\nu}(x) = \kappa\, t_{M,\mu\nu}(x)$ which can be integrated to a kind of Yang-Feldman equation [21]

$$U_{\mu\nu}(x) = U^{as}_{\mu\nu}(x) + \int d^4y\, \Delta^{as}_{\mu\nu;\alpha\beta}(x-y; M_r^2)\, t_M^{\alpha\beta}(y), \tag{7.19a}$$

$$\Delta^{as}_{\mu\nu;\alpha\beta}(x-y; M_r^2) = \left\{(\bar{\eta}_{\mu\alpha}\bar{\eta}_{\nu\beta} + \bar{\eta}_{\mu\beta}\bar{\eta}_{\nu\alpha}) - \frac{2}{3}\bar{\eta}_{\mu\nu}\bar{\eta}_{\alpha\beta}\right\} i\Delta^{as}(x-y; M_r^2) \tag{7.19b}$$

$$\to \left\{(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha}) - \frac{2}{3}\eta_{\mu\nu}\eta_{\alpha\beta}\right\} i\Delta^{as}(x-y; M_r^2), \tag{7.19c}$$

where $\bar{\eta}_{\mu\nu} \equiv \bar{\eta}(\partial, M_r^2)$ etc., and $U^{as}_{\mu\nu} = U^{in}_{\mu\nu}$ or $U^{out}_{\mu\nu}$ and corespondingly Δ^{as} is the retarded Δ_R or the advanced Δ_A. In (7.19) the replacement $\bar{\eta}_{\mu\nu} \to \eta_{\mu\nu}$ etc. follows from $\partial^\mu t_{M,\mu\nu} = (t_M)^\mu_\mu = 0$ and partial integration.

The physical Hilbert space $\mathcal{V}_{phys}$ consists of the totality of states $|f\rangle$ which satisfy the Gupta condition $\epsilon^{(+)}(x)|f\rangle = 0$. Representatives of equivalent classes are obtained by applying products of creation operators associated with the $U^{as}_{\mu\nu}$ field.

B. Massless $M = M_2 = 0$ case

In the limit $M, M_r \to 0$ the sum rules (7.12) reveal that

$$\lim_{M,M_r\to 0}(M_r^2/M^2)^n = Z\ (n=1,2) \Rightarrow Z \to 1, \tag{7.20}$$

which means the zero-mass limit $M = M_r \to 0$. Analyzing this limit for Eqn. (7.16) we note that with $(k_2 Z)^{-1}M_r^2 = M^2$ one obtains for the second term

$$(k_2 Z)^{-1}M_r^2\left(\eta_{\mu\nu} + \bar{\alpha}_{2r}\frac{\partial_\mu\partial_\nu}{M_r^2}\right) = M^2\eta_{\mu\nu} + \left(\frac{M_r}{M}\right)^2 \alpha_2\,\partial_\mu\partial_\nu \to \alpha_2\partial_\mu\partial_\nu.$$

Therefore, we obtain for the massless limit the equations

$$\Box h^{as}_{\mu\nu}(x) + 2\alpha_2\partial_\mu\partial_\nu\epsilon^{as}(x) = 0, \tag{7.21a}$$

$$\Box\epsilon^{as}(x) = 0,\ \partial^\mu h^{as}_{\mu\nu}(x) + \alpha_2\partial_\nu\epsilon^{as}(x) = 0, \tag{7.21b}$$

where we added the last equation from (7.1). Again, the derivatives in $\Delta^{as}_{\mu\nu;\alpha\beta}$ can be omitted. Next we take the limit $M \to 0$ which changes (7.19) to the massless form

$$\Delta^{as}_{\mu\nu;\alpha\beta}(x-y) \to \left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}\right)\ iD(x-y), \tag{7.22}$$

and changes (7.13) and (7.15) into the commutators

$$\left[h^{as}_{\mu\nu}(x), h^{as}_{\alpha\beta}(y)\right] = \left\{\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha}\right) - \eta_{\mu\nu}\eta_{\alpha\beta}\right\} iD(x-y), \tag{7.23a}$$

$$\left[h^{as}_{\mu\nu}(x), \epsilon^{as}(y)\right] = \frac{i}{2}\left(\eta_{\mu\nu}D(x-y) + \alpha_2\partial^\mu\partial^\nu\Box^{-2}E(x-y)\right), \tag{7.23b}$$

$$\left[\epsilon^{as}(x), \epsilon^{as}(y)\right] = -\frac{3i}{4}D(x-y), \tag{7.23c}$$

where $D(x) = \Delta(x;M^2)|_{M=0}$ and $E(x) = \Box D(x)$. Furthermore, in the massless limit the $U^{as}_{\mu\nu}(x)$ field now satisfies

$$\left[U^{as}_{\mu\nu}(x), U^{as}_{\alpha\beta}(y)\right] = \left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}\right)\ iD(x-y), \tag{7.24a}$$

$$\left[U^{as}_{\mu\nu}(x), \epsilon^{as}(y)\right] = 0, \tag{7.24b}$$

$$\Box U^{as}_{\mu\nu}(x) = 0,\ \partial^\mu U^{as}_{\mu\nu}(x) = 0, \tag{7.24c}$$

describing a massless graviton, demonstrating the smooth massless limit $M_2 \to 0$.

In Appendix A the quantization of the $\epsilon(x)$-field is described, and the propagator is derived in detail. Because of the (-)-sign in the last commutator in (7.19) it has a negative norm and so a Gupta-condition has to be imposed, similarly to that in the case of the Lorentz-gauge in QED. Then, the matrix elements between physical states of the equations (7.20) and of the commutators in (7.21) satisfy the classical equations, *i.e.* $\langle\Box h^{as}_{\mu\nu}(x)\rangle = 0, \langle\partial^\mu h_{\mu\nu}(x)\rangle = 0$.

Note that this auxiliary-field treatment for the massive and massless spin-2 meson is analogous to that for the vector-meson in Ref. [8] section 2.4, and distinct from the Proca-formalism used in [15]. The advantage is a smooth transition between the massive and massless case.

VIII. GRAVITON-MASS AND THE PERIHELIUM-PRECESSION OF PLANETS

We want to compute the finite-mass corrections to the massless spin-2 perihelium-precession of the planets. In this section, and henceforth, we denote the graviton mass by μ_G, the mass of the Sun by M, and the mass of Mercury by m.

The propagator for the (massive) $h^{\mu\nu}$-field, for $\lambda = 1$, see Eqn. (5.4), reads

$$\begin{aligned} D^{\mu\nu,\alpha\beta}(x;\mu_G^2)_F &= \frac{1}{2}\left(\eta^{\mu\alpha}\eta^{\nu\beta}+\eta^{\mu\beta}\eta^{\nu\alpha}-\eta^{\mu\nu}\eta^{\alpha\beta}\right)\,\Delta_F(x;\mu_G^2) \\ &\quad +\frac{1}{6}\eta^{\mu\nu}\eta^{\alpha\beta}\left[\Delta_F(x;\mu_G^2)-\Delta_F(x;M_\epsilon^2)\right] \qquad (8.1a) \\ &:= \Delta_{F,0}^{\mu\nu,\alpha\beta}(x;\mu_G^2)+\delta\Delta_F^{\mu\nu,\alpha\beta}(x;\mu_G^2), \qquad (8.1b) \end{aligned}$$

where $\Delta_{F,0}^{\mu\nu,\alpha\beta}(x)$ in the $\lim_{\mu_G\to 0}$ is the propagator for the massless spin-2 particle, and

$$\delta\Delta_{F,\mu\nu,\alpha\beta}(x;\mu_G^2) \equiv \Delta^{(S)}_{\mu\nu,\alpha\beta}(x;\mu_G^2)+\Delta^{(SG)}_{\mu\nu,\alpha\beta}(x;\mu_G^2), \qquad (8.2)$$

where

$$\begin{aligned} \Delta^{(S)}_{\mu\nu,\alpha\beta}(x;\mu_G^2) &= +\frac{1}{6}\eta^{\mu\nu}\eta^{\alpha\beta}\Delta_F(x;\mu_G^2), \qquad (8.3a) \\ \Delta^{(SG)}_{\mu\nu,\alpha\beta}(x;\mu_G^2) &= -\frac{1}{6}\eta^{\mu\nu}\eta^{\alpha\beta}\Delta_F(x;-2\mu_G^2) \qquad (8.3b) \end{aligned}$$

As noted before, in the massless $\lim_{\mu_G\to 0}$ *the propagator* $D_F^{\mu\nu,\alpha\beta}(x;\mu_G^2)$ *corresponds to a massless graviton. Therefore it gives the Einstein prediction for the perihelium precession.*
In Appendix B the smooth massless limit of the model is considered using the properties of the spin-2 polarization tensors for $\lim_{\mu_G\downarrow 0}$ and the conservation of the energy-momentum tensor $t_{M,\mu\nu}$.

A. Interaction Spin-2 Particles with a Scalar-field

In general-relativity [26, 27] the Lagrangian for a neutral scalar field, mass m, invariant under general coordinate transformations in a gravitational field described by the metrc $g_{\mu\nu}$, is given by

$$\mathcal{L}_S = \frac{1}{2}\sqrt{-g}\,\left(g^{\mu\nu}D_\mu\phi D_\nu\phi - m^2\phi^2\right), \qquad (8.4)$$

where $g = \det g_{\mu\nu}$, likewise $\eta = \det\eta_{\mu\nu} = -1$. In the "weak field" approximation we write

$$g_{\mu\nu} = \eta_{\mu\nu}+\kappa\, h_{\mu\nu},\;\; g^{\mu\nu} = \eta^{\mu\nu}-\kappa\, h_{\mu\nu}, \qquad (8.5)$$

where $\kappa \propto \sqrt{G}$ with G is the Newtonian gravitational constant. Note that the signs in (8.5) are consistent with $g^{\mu\nu}g_{\nu\lambda} = \delta^\mu_{\ \lambda}$.

In the following the lowering and raising of the indices is done with the $\eta_{\mu\nu} = \eta^{\mu\nu}$*-tensor.*

Using

$$g \approx \eta\,\left(1+\kappa\, h^\mu_\mu\right),\;\; \sqrt{-g}\approx 1+\frac{\kappa}{2}\,h^\mu_\mu, \qquad (8.6)$$

the scalar Lagrangian in the weak-field approximation is

$$\begin{aligned}
\mathcal{L}_S &= \frac{1}{2}\left(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right) \\
&\quad -\frac{\kappa}{2}h^{\mu\nu}\,\partial_\mu\phi\partial_\nu\phi + \frac{\kappa}{4}h^\lambda_\lambda\,\left(\eta^{\mu\nu}\partial_\mu\partial_\nu - m^2\phi^2\right) \\
&\equiv \mathcal{L}_S^{(0)} + \mathcal{L}_S^{int},
\end{aligned} \tag{8.7}$$

with

$$\begin{aligned}
\mathcal{L}_S^{int} &= -\frac{\kappa}{2}h^{\mu\nu}\,\partial_\mu\phi\partial_\nu\phi + \frac{\kappa}{4}h^\lambda_\lambda\,\left(\eta^{\mu\nu}\partial_\mu\partial_\nu - m^2\phi^2\right) \\
&= -\frac{\kappa}{2}h^{\mu\nu}\left[\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\,\left(\eta^{\alpha\beta}\partial_\alpha\partial_\beta - m^2\phi^2\right)\right] \\
&\equiv -\frac{\kappa}{2}h^{\mu\nu}\,t^{(S)}_{\mu\nu},
\end{aligned} \tag{8.8}$$

where

$$t^{(S)}_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \eta_{\mu\nu}\,\mathcal{L}_S^{(0)} \tag{8.9}$$

the energy-momentum tensor operator for the scalar field.
The momentum expansion for $h_{\mu\nu}(x)$ field reads

$$h_{\mu\nu}(x) = \int \frac{d^3k}{\sqrt{2\omega(k)(2\pi)^3}} \sum_{\lambda=1}^{5} e_{\mu\nu}(k,\lambda)\left[a(k,\lambda)e^{-ik\cdot x} + a^\dagger(k,\lambda)e^{+ik\cdot x}\right], \tag{8.10}$$

and for the (neutral) scalar field [28]

$$\phi(x) = \int \frac{d^3p}{\sqrt{2\omega(p)(2\pi)^3}}\left[a(p)e^{-ip\cdot x} + a^\dagger(p)e^{+ip\cdot x}\right], \tag{8.11}$$

with the commutation relation

$$\left[a(\mathbf{p}), a^\dagger(\mathbf{p}')\right] = \delta^3(\mathbf{p}-\mathbf{p}'). \tag{8.12}$$

For the one-particle scalar states we use the so-called "non-relativistic normalization":

$$|\mathbf{p}\rangle = a^\dagger(\mathbf{p})|0\rangle,\ \ \langle\mathbf{p}|\mathbf{p}'\rangle = \delta^3(\mathbf{p}-\mathbf{p}'). \tag{8.13}$$

Matrix elements for the energy-momentum operators, using **normal-ordering**, is

$$\langle p'|\,:\,t_{\alpha\beta}(x)\,:\,|p\rangle = \frac{\exp i(p'-p)\cdot x}{(2\pi)^3\sqrt{4E(p')E(p)}}\,\left[\left(p'_\alpha p_\beta + p'_\beta p_\alpha\right) - \eta_{\alpha\beta}\left(p'\cdot p - m^2\right)\right], \tag{8.14a}$$

$$\langle P'|\,:\,T_{\mu\nu}\,:\,|P\rangle = \frac{\exp i(P'-P)\cdot x}{(2\pi)^3\sqrt{4\mathcal{E}(P')\mathcal{E}(P)}}\,\left[\left(P'_\mu P_\nu + P'_\nu P_\mu\right) - \eta_{\mu\nu}\left(P'\cdot P - M^2\right)\right]. \tag{8.14b}$$

The (spin-0)-(spin-2) vertex is given by

$$\begin{aligned}
\langle p'| - i\int d^4x\,\mathcal{H}^{(S)}_{int}(x)|p,k\rangle &\equiv (2\pi)^4 i\delta^4(p'-p-k)\,e^{\mu\nu}(k,\lambda)\Gamma_{\mu\nu}(p',p)\,\cdot \\
&\quad \times\left[(2\pi)^9\;8E(p')E(p)\omega(k)\right]^{-1/2}
\end{aligned} \tag{8.15}$$

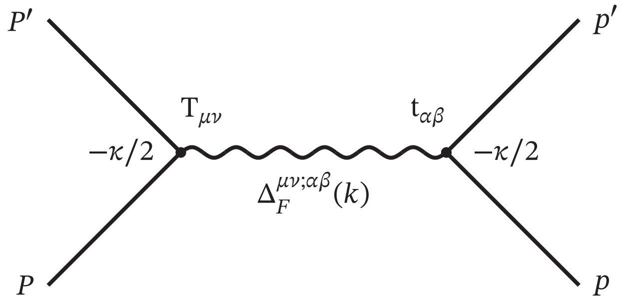

FIG. 2: Graviton-exchange between mass M and mass m.

Using

$$\langle 0|h^{\mu\nu}(x)|k,\lambda\rangle = \left[(2\pi)^3\, 2\omega(k)\right]^{-1/2}\, e^{\mu\nu}(k,\lambda)\, e^{-ik\cdot x}, \tag{8.16}$$

(8.15) leads to

$$\begin{aligned} \langle p'| - i\int d^4x\, \mathcal{H}^{(S)}_{int}(x)|p,k\rangle &= -i(\kappa/2)\, \left[(2\pi)^9\, 8\omega(p')\omega(p)\omega(k)\right]^{-1/2} \cdot \\ \times(2\pi)^4\delta^4(p'-p-k)\cdot e^{\mu\nu}(k,\lambda)&\Big\{\left(p_\mu p'_\nu + p'_\mu p_\nu\right) - \eta_{\mu\nu}(p'\cdot p - m^2)\Big\}, \end{aligned} \tag{8.17}$$

which leads to

$$\Gamma_{\mu\nu}(p',p) = -(\kappa/2)\,\Big[\left(p'_\mu p_\nu + p'_\nu p_\mu\right) - \eta_{\mu\nu}\left(p'\cdot p - m^2\right)\Big]. \tag{8.18}$$

We notice that $\Gamma_{\mu\nu}(p',p) = \Gamma_{\nu\mu}(p',p)$ and $(p'-p)^\mu\Gamma_{\mu\nu} = 0$. This gives for the matrix elements of the energy-momentum operators $T^{\mu\nu}$ and $t^{\alpha\beta}$

$$\widetilde{T}^{\mu\nu}(P',P) \;=\; \Big[\left(P'_\mu P_\nu + P'_\nu P_\mu\right) - \eta_{\mu\nu}\left(P'\cdot P - M^2\right)\Big], \tag{8.19a}$$

$$\widetilde{t}^{\alpha\beta}(p',p) \;=\; \Big[\left(p'_\alpha p_\beta + p'_\alpha p_\beta\right) - \eta_{\alpha\beta}\left(p'\cdot p - m^2\right)\Big]. \tag{8.19b}$$

Propagators:

$$\begin{aligned} \Delta^{(m)}_{\mu\nu;\alpha\beta}(k) &= \frac{P^{(m)}_{\mu\nu;\alpha\beta}(k)}{k^2-\mu_G^2+i\delta}, \\ P^{(m)}_{\mu\nu;\alpha\beta}(k) &= \frac{1}{2}\left(\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\mu\beta}\eta_{\nu\alpha}\right) - \frac{1}{3}\eta_{\mu\nu}\eta_{\alpha\beta}, \end{aligned} \tag{8.20a}$$

$$\begin{aligned} \Delta^{(0)}_{\mu\nu;\alpha\beta}(k) &= \frac{P^{(0)}_{\mu\nu;\alpha\beta}(k)}{k^2-\mu_G^2+i\delta}, \\ P^{(0)}_{\mu\nu;\alpha\beta}(k) &= \frac{1}{2}\left(\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\mu\beta}\eta_{\nu\alpha}-\eta_{\mu\nu}\eta_{\alpha\beta}\right). \end{aligned} \tag{8.20b}$$

For distinguishing the massless and massive case convenient is the common notation

$$P^{(a)}_{\mu\nu;\alpha\beta}(k) = \frac{1}{2}\left(\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\mu\beta}\eta_{\nu\alpha}\right) - a\;\eta_{\mu\nu}\eta_{\alpha\beta}, \tag{8.21}$$

where a=1/2 and 1/3 for the massless and massive case respectively.
The amplitude corresponding to the Feynman graph Fig. 2, for the Feynman rules see [29], chapter 14, is

$$\begin{aligned}
S^{(2)}_{fi} &= (+i)^2\,(\kappa^2/4)\,\mathcal{N}_f(P',p')\,\mathcal{N}_i(P,p)\int\frac{d^4k}{(2\pi)^4}(2\pi)^4\delta^4(P'-P+k)\,\cdot\\
&\quad\times(2\pi)^4\delta^4(p'-p-k)\cdot\langle P'|T^{\mu\nu}|P\rangle\,i\Delta^{(m)}_{\mu\nu;\alpha\beta}(k)\,\langle p'|t^{\alpha\beta}|p\rangle\\
&= -i(\kappa^2/4)\,(2\pi)^4\delta^4(P'+p'-P-p)\,\mathcal{N}_f(P',p')\,\mathcal{N}_i(P,p)\,\cdot\\
&\quad\times T^{\mu\nu}(P',P)\,\Delta^{(m)}_{\mu\nu;\alpha\beta}(k)\,t^{\alpha\beta}(p',p)\\
&\equiv -i(2\pi)^4\delta^4(P'+p'-P-p)\,\mathcal{N}_f\,M^{(2)}_{fi}\,\mathcal{N}_i \qquad (8.22)
\end{aligned}$$

with $k=p'-p=P-P'$, and

$$\mathcal{N}_i(P,p)=\left[(2\pi)^6\,\mathcal{E}(P)\,E(p)\right]^{-1/2},\ \ \mathcal{N}_f(P',p')=\left[(2\pi)^6\,\mathcal{E}(P')\,E(p')\right]^{-1/2}. \qquad (8.23)$$

Here $E(p)=\omega(p), \mathcal{E}(P)=\omega(P)$ etc.
With $a=1/3$ and $a=1/2$ for the massive (m) and massless (0) case respectively, one gets for the invariant amplitude the expression

$$\begin{aligned}
&M^{(2)}_{fi}(P',p';P,p)=\frac{1}{4}\kappa^2\,\langle P'|T^{\mu\nu}|P\rangle\,\Delta^{(m,0)}_{\mu\nu;\alpha\beta}(k)\,\langle p'|t^{\alpha\beta}|p\rangle=\frac{1}{4}\kappa^2\,\cdot\\
&\times\left[(P'_\mu P_\nu+P'_\nu P_\mu)-\eta_{\mu\nu}(P'\cdot P-M^2)\right]\cdot\left\{\frac{1}{2}\left(\eta^{\mu\alpha}\eta^{\nu\beta}+\eta^{\mu\beta}\eta^{\nu\alpha}\right)-a\,\eta^{\mu\nu}\eta^{\alpha\beta}\right\}\cdot\\
&\times\left[\left(p'_\alpha p_\beta+p'_\beta p_\alpha\right)-\eta_{\alpha\beta}(p'\cdot p-m^2)\right]\cdot\left[k^2-\mu_G^2+i\delta\right]^{-1}=\\
&\frac{1}{2}\kappa^2\,\Big\{(P'\cdot p')(P\cdot p)+(P'\cdot p)(P\cdot p')-2a\,(P'\cdot P)(p'\cdot p)\\
&-2(4a-1)\,M^2m^2+(4a-1)\left[m^2\,(P'\cdot P)+M^2\,(p'\cdot p)\right]\Big\}\cdot\left[k^2-\mu_G^2+i\delta\right]^{-1}. \qquad (8.24)
\end{aligned}$$

Now, $k=p'-p=P-P'$ which gives with $P^2=P'^2=M^2$ and $p^2=p'^2=m^2$,

$$\begin{aligned}
&k^2=(P'-P)^2=2M^2-2P'\cdot P=(p'-p)^2=2m^2-2p'\cdot p,\\
&P'\cdot P=M^2-\frac{1}{2}k^2,\ \ p'\cdot p=m^2-\frac{1}{2}k^2,
\end{aligned}$$

and similar expressions for $P'\cdot p, P\cdot p'$ in the Mandelstam variables $s=(P+p)^2=(P'+p')^2$ and $u=(P-p')^2=(P'-p)^2$. In the C.M.-system, $\mathbf{P}=-\mathbf{p}$ and $\mathbf{P}'=-\mathbf{p}'$, and $\mathbf{k}=\mathbf{p}'-\mathbf{p}$, $\mathbf{q}=(\mathbf{p}'+\mathbf{p})/2$.

$$\mathbf{p}=\mathbf{q}-\frac{1}{2}\mathbf{k},\ \ \mathbf{p}'=\mathbf{q}+\frac{1}{2}\mathbf{k}. \qquad (8.25)$$

The scalar products in the C.M.-system read, taking the external particles on the energy shell i.e. $\mathbf{p}^2=\mathbf{p}'^2$, giving $k^2=-\mathbf{k}^2$, $\mathbf{q}\cdot\mathbf{k}=0$. Evaluating the scalar products in (8.24) in the C.M.-system, we expand in $1/Mm$. Then, we obtain the leading terms

$$\begin{aligned}
M^{(2)}_{fi}(P',p';P,p)\ &\approx\ -\frac{1}{4}\kappa^2\,(2Mm)^2\left\{(1-a)+\frac{(M+m)^2}{M^2m^2}\,(\mathbf{q}^2+\frac{1}{4}\mathbf{k}^2)\right.\\
&\left.+\frac{(2a-1)(M^2+m^2)-2Mm}{4M^2m^2}\,\mathbf{k}^2\right\}\,\left(\mathbf{k}^2+\mu_G^2-i\delta\right)^{-1}. \qquad (8.26)
\end{aligned}$$

In Appendix C, the connection between the invariant amplitude $M^{(2)}_{fi}$ and the potential in the Lippmann-Schwinger (or Schrödinger) equation is given by (C18)

$$\begin{aligned}\mathcal{V}(p_f, p_i) &\approx \frac{\pi}{2Mm}\left(1-\frac{M^2+m^2}{4M^2m^2}(\mathbf{q}^2+\mathbf{k}^2/4)\right)M^{(2)}(p_f,p_i;W)\\ &\approx -\frac{\pi}{2}[\kappa^2\, Mm]\left\{(1-a)+\frac{2a(M^2+m^2)-(M+m)^2}{4M^2m^2}\,\mathbf{k}^2\right.\\ &\left.+\left[\frac{(M+m)^2}{M^2m^2}-(1-a)\frac{M^2+m^2}{4M^2m^2}\right](\mathbf{q}^2+\frac{1}{4}\mathbf{k}^2)\right\}\cdot\\ &\times\left(\mathbf{k}^2+\mu_G^2-i\delta\right)^{-1}. \end{aligned} \tag{8.27}$$

1. Massless case, a=1/2: From (8.27) the on energy-shell potential becomes

$$\begin{aligned}\mathcal{V}^{(0)}_{fi}(P',p';P,p) &= -\frac{\pi}{4}[\kappa^2\, Mm]\left\{1+\frac{7M^2+16Mm+7m^2}{4M^2m^2}(\mathbf{q}^2+\frac{1}{4}\mathbf{k}^2)\right.\\ &\left.-\frac{\mathbf{k}^2}{Mm}\right\}\cdot\left[\mathbf{k}^2+i\delta\right]^{-1}. \end{aligned} \tag{8.28}$$

The potential in momentum space from the leading term is

$$\mathcal{V}^{(0)}(r) = \int\frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, V^{(0)}(\mathbf{p}',\mathbf{p}) = -\frac{\kappa^2 Mm}{16}\,\frac{1}{r} \Rightarrow -\frac{GMm}{r}, \tag{8.29}$$

i.e. equating this potential to the Newton potential we get, using units $\hbar = c = 1$,

$$\kappa/4 = \sqrt{G} = 1.616\,\times\,10^{-33}\,cm\ \ (\textit{\textbf{Planck length}}). \tag{8.30}$$

2. Massive case, a=1/3: From (8.27) the on energy-shell potential becomes

$$\begin{aligned}\mathcal{V}^{(m)}_{fi}(p_f,p_i) &= -\frac{2\pi}{6}[\kappa^2\, Mm]\left\{1+\frac{5M^2+12Mm+5m^2}{4M^2m^2}(\mathbf{q}^2+\frac{1}{4}\mathbf{k}^2)\right.\\ &\left.-\frac{M^2+6Mm+m^2}{2Mm}\frac{\mathbf{k}^2}{4Mm}\right\}\cdot\left[\mathbf{k}^2+\mu_G^2\right]^{-1} \end{aligned} \tag{8.31}$$

In the C.M.-system we write $V^{(2)}(P',p';P,p)\equiv(2\pi)^{-6}V^{(m)}(\mathbf{p}',\mathbf{p})$, and the configuration space potential is given by

$$(\mathbf{x}'|V|\mathbf{x}) = \int\frac{d^3k}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\, e^{i\mathbf{k}\cdot(\mathbf{x}'+\mathbf{x})/2}\, e^{i\mathbf{q}\cdot(\mathbf{x}'-\mathbf{x})}\, V(\mathbf{k},\mathbf{q}). \tag{8.32}$$

From (8.31) one gets, neglecting for the moment the $\mathbf{k}^2$ and $\mathbf{q}^2$ terms, the C.M.-system local potential becomes

$$\mathcal{V}^{(m)}(r) = \int\frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, V^{(m)}(\mathbf{p}',\mathbf{p}) = -\frac{\kappa^2 Mm}{12}\,\frac{e^{-\mu_G r}}{r}. \tag{8.33}$$

Note that for very small μ_G we have

$$\mathcal{V}^{(m)}(r)\approx\frac{4}{3}\,\mathcal{V}^{(0)}(r), \tag{8.34}$$

i.e. the massless limit of the massive potential is off by a factor 4/3, a well known fact [1].

B. Perihelium Precession

1. Massless case: We analyze the $\mathbf{k}^2$- and $\mathbf{q}^2$-term in the momentum space potential (8.28), which we write as

$$V^{(0)}(\mathbf{p}', \mathbf{p}) = -(4\pi GMm) \left\{1 - \frac{\mathbf{k}^2}{Mm} + \frac{7(M+m)^2 + 2Mm}{4M^2m^2}(\mathbf{q}^2 + \frac{1}{4}\mathbf{k}^2)\right\} \cdot \times \left[\mathbf{k}^2 + \mu_G^2 - i\delta\right]^{-1}. \tag{8.35}$$

Here, we introduced the graviton mass μ_G because we want to analyze the massless limit. So, in (8.35) the massless graviton projection operator is used in the propagator.

(a) The central term in (8.35) gives the Newtonian potential

$$\mathcal{V}^{(0)}(r) = -[GMm]\,\frac{e^{-\mu_G r}}{r} \text{ with } \mu_G = 0. \tag{8.36}$$

(b) The $\mathbf{k}^2$-terms in (8.31) and (8.35) give terms with $-4\pi\delta^3(\mathbf{r})$ which in the planetary motion do not contribute and hence can be dropped.

(c) The Fourier transformation to configuration space of the non-local $(\mathbf{q}^2 + \mathbf{k}^2/4)$-term is [12]

$$(\mathbf{r}|\mathcal{V}^{(1)}|\psi) = -\frac{7(M+m)^2}{8M^2m^2}\left(\boldsymbol{\nabla}^2\mathcal{V}^{(0)}(r) + \mathcal{V}^{(0)}(r)\boldsymbol{\nabla}^2\right)\psi(\mathbf{r}). \tag{8.37}$$

The Schrödinger equation reads

$$\left(-\frac{\boldsymbol{\nabla}^2}{2m_{red}} + \mathcal{V}\right)\psi = E\,\psi \;\left(m_{red} = \frac{Mm}{M+m}\right),$$

and gives the possibility of the replacement $\boldsymbol{\nabla}^2 \rightarrow 2m_{red}(\mathcal{V}_C - E)$, where $\mathcal{V}_C$ is the total central potential. (The spin-spin, tensor and spin-orbit potentials in $\mathcal{V}$ are of order $1/Mm$ and can be neglected.) For (very) small μ_G we have $\mathcal{V}_C = -A\,GMm/r = A\,\mathcal{V}^{(0)}$. From $\left[\boldsymbol{\nabla}^2, \mathcal{V}_C(r)\right]\psi(r) = \left[[\boldsymbol{\nabla}^2\mathcal{V}_C(r)] + 2\boldsymbol{\nabla}\mathcal{V}_C(r)\cdot\boldsymbol{\nabla}\right]\psi(r)$ neglecting $\boldsymbol{\nabla}\psi$ [31] and the $\boldsymbol{\nabla}^2$-term, which gives $\approx \delta^3(\mathbf{r})$ contribution, the contribution of the non-local term gives the correction to the Newton-potential

$$\begin{aligned} \mathcal{V}^{(1)}(r) &\approx -2m_{red}\frac{7(M+m)^2}{4M^2m^2}\,\mathcal{V}^{(0)}(r)\left[\mathcal{V}_C(r) - E\right] \\ &= -\frac{7}{2m}\left(1 + \frac{m}{M}\right)\frac{GMm}{r}\left[A\frac{GMm}{r} + E\right] \\ &\Rightarrow -\frac{7A}{2m}\left[\mathcal{V}^{(0)}\right]^2 m \sim 1/r^2 \;\; (M \gg m). \end{aligned} \tag{8.38}$$

---

[12] A potential term $V(\mathbf{k}, \mathbf{q}) = \widetilde{V}_{n.l.}(\mathbf{k})\left(\mathbf{q}^2 + \mathbf{k}^2/4\right)$ gives in coordinate space, see [30] Eqn. (11),

$$\mathcal{V}^{(1)}(r) = -\frac{1}{2}\left(\boldsymbol{\nabla}^2\mathcal{V}_{n.l.}(r) + \mathcal{V}_{n.l.}(r)\boldsymbol{\nabla}^2\right).$$

In the last step we used that in a planetary orbit E= constant, and the E-term in (8.38) gives a correction to the Newtonian potential which means a small modification of the orbit. Henceforth, being interested here only in the $1/r^2$ potential, we omit the E-term.
We see that the non-local potential gives a $1/r^2$ correction leading to a **perihelium-precession** which is $7/6\times$ Einstein's result! This agrees with the treatment of Schwinger [20, 32]. The remaining $-1/6$ comes from the change in the gravitional field energy due to the presence of the planet, see for discussion [20, 32].

2. Massive case: Following the same steps as for the massless case, taking into account that $\mathcal{V}_C^{(m)} = (4/3)\mathcal{V}^{(0)}(r)$ and the non-local term with 5/3 instead of 7/4, we obtain

$$\mathcal{V}^{(1)}(r) \Rightarrow -\frac{40A}{9m}\left[\mathcal{V}^{(0)}\right]^2. \tag{8.39}$$

Below we show that the contributions from the scalar (section IX A) and ghost (section IX B) to the non-local potential cancel each other in the massless limit. Therefore, the total result for the $1/r^2$ correction to the Newtonian potential from the $h^{\mu\nu}$-field is given by (8.38). Furthermore, corrections to the perihelium precession for a finite mass μ_G turn out proportional to μ_G^2 and are tiny, see section X.
We see that the non-local potential gives a $1/r^2$ correction leading to a **perihelium-precession** which is $7/6\times$ Einstein's result! This agrees with the treatment of Schwinger [20, 32]. The remaining $-1/6$ comes from the change in the gravitional field energy due to the presence of the planet, see for discussion [20, 32].

See Appendix C, under point 4, for alternative treatments and a more complete discussion.

IX. SCALAR CONTRIBUTIONS PERIHELIUM PRECESSION

In this section we derive the contribution to the perihelium precession from the scalar and scalar-ghost terms in the $h_{\mu\nu}$-propagator $\Delta_F^{\mu\nu,\alpha\beta}(x;\mu_G^2)$ (8.2).

A. Scalar-exchange: Perihelium Precession

The amplitude corresponding to the Feynman graph Fig. 2, for scalar-exchange is, in analogy with Eqn. (8.22), given by

$$\begin{aligned} S_{fi}^{(2)} &= (+i)^2\,(\kappa^2/4)\,\mathcal{N}_f(P',p')\,\mathcal{N}_i(P,p)\int\frac{d^4k}{(2\pi)^4}(2\pi)^4\delta^4(P'-P+k)\,\cdot \\ &\quad\times(2\pi)^4\delta^4(p'-p-k)\cdot\langle P'|T^{\mu\nu}|P\rangle\, i\Delta^{(S)}_{\mu\nu;\alpha\beta}(k)\,\langle p'|t^{\alpha\beta}|p\rangle \\ &= -i(\kappa^2/4)\,(2\pi)^4\delta^4(P'+p'-P-p)\,T^{\mu\nu}(P',P)\,\Delta^{(S)}_{\mu\nu;\alpha\beta}(k)\,t^{\alpha\beta}(p',p) \\ &\equiv -i(2\pi)^4\delta^4(P'+p'-P-p)\,\mathcal{N}_f\,M_{fi}^{(S)}\,\mathcal{N}_i \end{aligned} \tag{9.1}$$

where

$$\Delta^{(S)}_{\mu\nu;\alpha\beta}(k) \;=\; \frac{P^{(S)}_{\mu\nu;\alpha\beta}(k)}{k^2-\mu_G^2+i\delta},\;\; P^{(S)}_{\mu\nu;\alpha\beta}(k)=\frac{1}{6}\eta_{\mu\nu}\eta_{\alpha\beta}. \tag{9.2}$$

Up to terms of order $1/M^2$ and $1/m^2$, taking only the terms proportional to the parameter a in the expression (8.27) and putting $a=-1/6$, one finds

$$M^{(2)}_{fi}(P',p';P,p) \;\approx\; -\frac{1}{24}\kappa^2\,(2Mm)^2\left\{1-\frac{(M^2+m^2)-Mm}{2M^2m^2}\,\mathbf{k}^2\right\}\left(\mathbf{k}^2+\mu_G^2-i\delta\right)^{-1}. \tag{9.3}$$

In Appendix C, the connection between the invariant amplitude $M^{(2)}_{fi}$ and the potential in the Lippmann-Schwinger (or Schrödinger) equation is given by (C18)

$$\begin{aligned}
\mathcal{V}^{(S)}(p_f,p_i) \;&\approx\; \frac{\pi}{2Mm}\left(1-\frac{M^2+m^2}{4M^2m^2}(\mathbf{q}^2+\mathbf{k}^2/4)\right)M^{(S)}(p_f,p_i;W)\\
&\approx\; -\frac{\pi}{12}[\kappa^2\,Mm]\left\{1-\frac{(M^2+m^2)-Mm}{2M^2m^2}\,\mathbf{k}^2\right.\\
&\qquad\left.-\frac{M^2+m^2}{4M^2m^2}(\mathbf{q}^2+\frac{1}{4}\mathbf{k}^2)\right\}\cdot\left(\mathbf{k}^2+\mu_G^2-i\delta\right)^{-1}.
\end{aligned} \tag{9.4}$$

The central potential in the CM-system is, with $\kappa^2=16G$,

$$\mathcal{V}^S_C(r) \;=\; -\frac{\pi}{12}\left[\kappa^2\,Mm\right]\int\frac{d^3k}{(2\pi)^3}\,\frac{e^{(i\mathbf{k}\cdot\mathbf{r})}}{\mathbf{k}^2+\mu_G^2} = -\frac{1}{3}\,[GMm]\,\frac{e^{-\mu_G r}}{r}. \tag{9.5}$$

The contribution of the non-local term gives the correction to the Newtonian potential

$$\begin{aligned}
\mathcal{V}^{(1)}(r) \;&\approx\; +2m_{red}\frac{M^2+m^2}{4M^2m^2}\;\mathcal{V}^S_C(r)\,[\mathcal{V}_C(r)-E]\\
&\Rightarrow\; +\frac{A}{2m}\left[\mathcal{V}^{(0)}\right]^2 = +\frac{A}{18m}\,[GMm]^2\;\frac{e^{-2\mu r}}{r^2}\sim 1/r^2\;\;(M\gg m).
\end{aligned} \tag{9.6}$$

B. Scalar-ghost-exchange: Perihelium Precession

In Appendix D the details of the scalar interaction with the imaginary-ghost field is analyzed and worked out in detail, which results are employed in this section.
The amplitude corresponding to the Feynman graph Fig. 2, for scalar-ghost-exchange is, in analogy with Eqn. (8.22), given by

$$\begin{aligned}
S^{(2)}_{fi} \;&=\; (+i)^2\,(\kappa^2/4)\,\mathcal{N}_f(P',p')\,\mathcal{N}_i(P,p)\,\int_\Gamma\frac{d^4k}{(2\pi)^4}(2\pi)^4\delta^4(P'-P+k)\;\cdot\\
&\quad\times(2\pi)^4\delta^4(p'-p-k)\cdot\langle P'|T^{\mu\nu}|P\rangle\,i\Delta^{(SG)}_{\mu\nu;\alpha\beta}(k)\,\langle p'|t^{\alpha\beta}|p\rangle\\
&=\; -i(\kappa^2/4)\,(2\pi)^4\delta^4(P'+p'-P-p)\,T^{\mu\nu}(P',P)\,\Delta^{(SG)}_{\mu\nu;\alpha\beta}(k)\,t^{\alpha\beta}(p',p)\\
&\equiv\; -i(2\pi)^4\delta^4(P'+p'-P-p)\,\mathcal{N}_f\,M^{(SG)}_{fi}\,\mathcal{N}_i
\end{aligned} \tag{9.7}$$

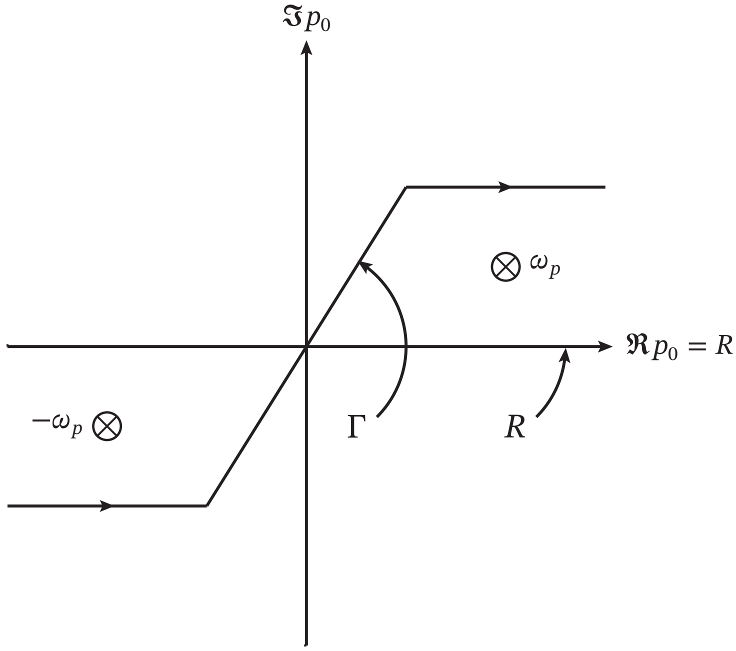

FIG. 3: Complex p_0-plane contour $\Gamma = R + \delta(-\omega_p) - \delta(\omega_p)$. Here, δ-operator is Leray's coboundary operator.

where

$$\Delta^{(SG)}_{\mu\nu;\alpha\beta}(k) = -\frac{1}{6}\eta_{\mu\nu}\eta_{\alpha\beta}\,\widetilde{\Delta}_F(k;-\mu^2). \tag{9.8}$$

Here, the imaginary ghost-mass is $-\mu^2 \equiv M_\epsilon^2 = -2\mu_G^2$. The proper integration contour Γ, see Fig. 3, in the complex k_0-plane is given as [6] $\Gamma = R - \delta(\omega_k) + \delta(-\omega_k)$, where $\delta(\pm\omega_k)$ denotes a *counterclockwise circle* around the poles at $\pm\omega_k$ [25]. (Note that for a real mass M_ϵ the Γ-contour becomes the usual countour for the Feynman propagator C_F.)
The Feynman propagator function in (9.8) is

$$\widetilde{\Delta}(k;M_\epsilon^2) = \frac{1}{2}\left[\widetilde{\Delta}^{(1)}(k;M_\epsilon^2) + \widetilde{\Delta}^{(2)}(k;M_\epsilon^2)\right], \tag{9.9a}$$

$$\widetilde{\Delta}^{(1)}(k;M_\epsilon^2) = \frac{1}{k^2 - M_\epsilon^2}, \quad \int \frac{d^4p}{(2\pi)^4} \to \int \frac{d^3k}{(2\pi)^3} \int_\Gamma \frac{dk_0}{(2\pi)}, \tag{9.9b}$$

$$\widetilde{\Delta}^{(2)}(k;M_\epsilon^2) = \frac{1}{k^2 - M_\epsilon^{*2}}, \quad \int \frac{d^4p}{(2\pi)^4} \to \int \frac{d^3k}{(2\pi)^3} \int_R \frac{dk_0}{(2\pi)}. \tag{9.9c}$$

Up to terms of order $1/M^2$ and $1/m^2$, taking only the terms proportional to the parameter a in the expression (8.26) and putting $a = +1/6$, one finds

$$M^{(SG)}_{fi}(P',p';P,p) \approx +\frac{1}{24}\kappa^2\,(2Mm)^2\left\{1 - \frac{(M^2+m^2) - Mm}{2M^2m^2}\,\mathbf{k}^2\right\}\mathcal{P}\frac{1}{\mathbf{k}^2 - \mu^2}. \tag{9.10}$$

Here, we used $\widetilde{\Delta}(k, M_\epsilon^2) = \mathcal{P}(\mathbf{k}^2 - \mu^2)^{-1}$, which is derived explicitly in Appendix A 1.
In Appendix C, the connection between the invariant amplitude $M^{(2)}_{fi}$ and the potential in the

Lippmann-Schwinger (or Schrödinger) equation is given by (C18)

$$\begin{aligned}\mathcal{V}^{(SG)}(p_f,p_i) &\approx \frac{\pi}{2Mm}\left(1-\frac{M^2+m^2}{4M^2m^2}(\mathbf{q}^2+\mathbf{k}^2/4)\right)M^{(SG)}(p_f,p_i;W)\\ &\approx +\frac{\pi}{12}[\kappa^2\,Mm]\left\{1-\frac{(M^2+m^2)-Mm}{2M^2m^2}\,\mathbf{k}^2\right.\\ &\qquad \left.-\frac{M^2+m^2}{4M^2m^2}(\mathbf{q}^2+\frac{1}{4}\mathbf{k}^2)\right\}\cdot\mathcal{P}\frac{1}{\mathbf{k}^2-\mu^2}.\end{aligned} \tag{9.11}$$

The central potential in the CM-system is

$$\mathcal{V}_C^{SG}(r) = +\frac{\pi}{12}[\kappa^2\,Mm]\int\frac{d^3k}{(2\pi)^3}\,\mathcal{P}\,\frac{e^{(i\mathbf{k}\cdot\mathbf{r})}}{\mathbf{k}^2-\mu^2} = +\frac{1}{3}\,[GMm]\,\frac{\cos(\mu r)}{r}. \tag{9.12}$$

The contribution of the non-local term gives the correction to the Newtonian potential

$$\begin{aligned}\mathcal{V}^{(1)}(r) &\approx +2m_{red}\frac{M^2+m^2}{4M^2m^2}\,\mathcal{V}_C^{(SG)}(r)\,[\mathcal{V}_C(r)-E]\\ &\Rightarrow -\frac{A}{2m}\left[\mathcal{V}_C^{(SG)}\mathcal{V}^{(0)}\right] = +\frac{A}{18m}\,[GMm]^2\,\frac{\cos(\mu r)}{r^2}\sim 1/r^2\ \ (M\gg m).\end{aligned} \tag{9.13}$$

The total result for the non-local $1/r^2$-potential from massive and scalar-ghost is

$$\mathcal{V}^{(1)}(total) = \mathcal{V}_m^{(1)}+\mathcal{V}_{SG}^{(1)}\to +\frac{7A}{2m}\left[\mathcal{V}^{(0)}\right]^2 \tag{9.14}$$

for $\lim\mu_G\to 0$. *Below we will show that the correction for the finite graviton mass* $\sim\mu_G^2$, *which is very small.*

X. RESULTS PERIHELIUM PRECESSION

Taking for the definition of the gravitational constant that which occurs in the massless case $\mu_G=0$, i.e. $\kappa^2=16G$, the gravitational potential in momentum space is of the form

$$\begin{aligned}\mathcal{V}(p_f,p_i) &= -4\pi\,[GMm]\left\{A+B\mathbf{k}^2/m^2-C\left(\mathbf{q}^2+\frac{1}{4}\mathbf{k}^2\right)/m^2\right\}\cdot\\ &\quad\times(2\pi)^{-6}\left(\mathbf{k}^2+\mu_G^2-i\delta\right).\end{aligned} \tag{10.1}$$

Here A,B, and C contain the total contributions, *i.e.* $A=\sum_i A_i$ etc. where A_i,B_i,C_i come from the individual contributions. In coordinate space we get for the central and non-local potential

$$\mathcal{V}^{(0)}(r) \equiv -[GMm]\,\frac{e^{-\mu_G r}}{r}\,,\ \mathcal{V}_C(r)=A\,\mathcal{V}^{(0)}(r), \tag{10.2a}$$

$$\left(\mathbf{r}|\,\mathcal{V}^{(1)}|\psi\right) = +\frac{1}{2}\,C\left(\boldsymbol{\nabla}^2\mathcal{V}^{(0)}(r)+\mathcal{V}^{(0)}(r)\boldsymbol{\nabla}^2\right)/m^2. \tag{10.2b}$$

Making the approximation $\boldsymbol{\nabla}\psi \approx 0$, and $\boldsymbol{\nabla}^2 \mathcal{V}^{(0)} \sim \delta^3(\mathbf{r}) \to 0$ for the planetary motion, leads to $[\boldsymbol{\nabla}^2, \mathcal{V}^{(0)}(r)] \approx 0$. Then, as described above, using the Schrödinger equation one makes the replacement $\boldsymbol{\nabla}^2 \to 2m_{red}(\mathcal{V}_C - E)$ where $m_{red} = m$. Then, we arrive at the correction to the Newtonian potential, see [30] Eqn. (33),

$$\begin{aligned} \mathcal{V}^{(1)}(r) &\approx 2m_{red}\,\left(C/m^2\right)\,\mathcal{V}^{(0)}(r)\left[\mathcal{V}_C(r) - E\right] \\ &\Rightarrow 2CA\,\frac{\left[\mathcal{V}^{(0)}\right]^2}{m} = 2\left(AC/m\right)\,[GMm]^2/r^2\;(M \gg m). \end{aligned} \tag{10.3}$$

As explained above the contribution of the $\mathbf{k}^2$-term can be neglected in the central potential.

Exchange	Propagator	A	B	C
I: Massless	$\Delta^{(0)}_{\mu\nu,\alpha\beta}$	1	$\frac{m}{M}$	$-\frac{7}{4}$
II: Massive	$\Delta^{(m)}_{\mu\nu,\alpha\beta}$	$\frac{4}{3}$	$\frac{m}{M}-\frac{1}{6}$	$-\frac{5}{3}$
III: Ghost	$\Delta^{(SG)}_{\mu\nu,\alpha\beta}$	$-\frac{1}{3}$	$+\frac{1}{6}$	$-\frac{1}{12}$
IV: "Scalar"	$\Delta^{(S)}_{\mu\nu,\alpha\beta}$	$\frac{1}{3}$	$-\frac{1}{6}$	$+\frac{1}{12}$

TABLE I: Coefficients A, B, C for the different exchange types

In Table X the coefficients A,B, and C are listed for the exchanges calculated in this paper. The propagators are explicitly defined in Eqn's (8.1-8.3). Adding the contributions II and III gives I, as expected.

The results in Table X show that in the $\lim_{\mu_G \to 0}$ the $h^{\mu\nu}$ propagator leads to the masless graviton contribution for the perihelium precession, due to the combination of the massive spin-2 and ghost contributions: from Table X we obtain $AC = (4/3 - 1/3) * (-5/3 - 1/12) = -7/4$, which corresponds to the value for the massless propagator. Then, $\mathcal{V}^{(1)}(r) = 2AC[\mathcal{V}^{(0)}(r)]^2/m = -(7/2m)[\mathcal{V}^{(0)}(r)]^2$.

In this treatment using the $\mathcal{V}^{(1)}$-interaction the change in the gravitational field energy due to the presence of the sun and the planet is not included. In Appendix E we review the derivation of this effect, which give a contribution $\left[\mathcal{V}^{(0)}\right]^2/2m$. Including this we get in total $\mathcal{V}^{(1)} = -(3/m)\left[\mathcal{V}^{(0)}\right]^2$, which agrees with Einstein's result.

Remark: *We note that this* $1/r^2$ *correction to the Newton potential does agree with the* $-(3V^2/m+V^2/2m)$*-correction in Ref. [20] below formula (52). In [20, 32] the* $-3V^2/m$ *comes from*

$$\begin{aligned} E_{int} &= -\frac{2GM}{r}\left(\sqrt{p^2+m^2}-\frac{1}{2}\frac{m^2}{\sqrt{p^2+m^2}}\right) \\ &\approx -\frac{GMm}{r}\left[1+\left(1+\frac{1}{2}\right)\frac{p^2}{m}\right], \end{aligned}$$

and $-V^2/2m$ *comes from the relativistic correction to the kinetic energy of the planet:* $T \to \sqrt{p^2+m^2}-m$ *replacing* T *by* $T-T^2/2m \sim T-(E-V)^2/2m$.

The planetary equation for the massless graviton, with the inclusion of the gravitational field energy between the planet and the sun reads, see [32] Eqns (2-4.55)-(2-4.60),

$$\frac{d^2u_0}{d\varphi^2}+u_0 = -\frac{1}{L_1^2}\frac{d}{du}\frac{V_{eff}^{(0)}}{m}, \tag{10.4a}$$

$$V_{eff}^{(0)} = V-3V^2/m,\ V=-GMm/r, \tag{10.4b}$$

where $u=1/R\equiv 1/d$ and L_1 is the angular momentum per unit planetary mass. This gives (in units c=1)

$$\frac{d^2u_0}{d\varphi^2}+\left(1-\frac{6G^2M^2}{L_1^2}\right)u_0=\frac{GM}{L_1^2}, \tag{10.5}$$

leading to the non-newtonian correction to the perihelium precession angle

$$\Delta\varphi_E = 6\pi G^2M^2/L_1)^2 = 6\pi\left(1+\frac{m}{M}\right)GM/L, \tag{10.6}$$

where $L^{-1}=(1/r_+ + 1/r_-)/2=\mu GMm/J^2$. Here, $r_\pm$ are the apohelium and perihelium distances. The connection with Einstein's result [26] is given by the relation $GM/L_1 = 2\pi(a/T)(1-e^2)^{-1/2}$, where a is the semimajor axis, T is the period, and e is the eccentricity.
In Table X the results for the $\Delta\varphi_E$ correction to the perihelium precession per century are listed for Icarus, Mercure, Venus, Earth, and Mars. The astronomical data in the next two tables are taken from Ref. [33].

A. Finite-mass and Ghost correction Perihelium-precession

The finite-mass correction are due to the differences

$$\begin{aligned} \delta^{(1)}\Delta_F^{\mu\nu,\alpha\beta}(x-y;\mu_G^2) &= \frac{1}{2}\left(\eta^{\mu\alpha}\eta^{\nu\beta}+\eta^{\mu\beta}\eta^{\nu\alpha}-\eta^{\mu\nu}\eta^{\alpha\beta}\right)\cdot \\ &\quad \times\left[\Delta_F(x-y;\mu_G^2)-\Delta_F(x-y;\mu_G^2=0)\right], \end{aligned} \tag{10.7a}$$

$$\delta^{(2)}\Delta_F^{\mu\nu,\alpha\beta}(x-y;\mu_G^2) = \frac{1}{6}\eta^{\mu\nu}\eta^{\nu\alpha}\left[\Delta_F^{(S)}(x-y;\mu_G^2)-\Delta_F^{(SG)}(x-y;\mu_G^2=0)\right]. \tag{10.7b}$$

Planet	Icarus	Mercury	Venus	Earth	Mars
Mass	$0.245 10^{-12}$	0.055	0.820	1.000	0.110
r_{min}(AU)	0.186	0.307	0.717	0.981	1.524
ϵ	0.827	0.206	0.0068	0.0167	0.0915
Period	409	87.97	224.70	365.26	686.98
J	0.157×10^{-12}	0.091	1.80	2.70	0.35
L_1	0.48	0.111	0.154	0.182	0.235
n	89.3	415.2	162.5	100.0	53.17
$n\Delta\varphi$	9.8 ± 0.8	43.1 ± 0.5	8.4 ± 4.8	5.0 ± 1.2	1.52
GRT/LRGT-AF	10.0	43.0	8.6	3.8	1.63

TABLE II: The Solar System. Mass planet m_{pl} in earth masses, $r_{min}(AU)$, ϵ eccentricity, $J = 2\pi m_{pl} r_{min}^2/T$: orbital angular momentum (units $10^{40} kg\ m^2/s$), L_1: angular momentum per unit mass (units $10^{16} m^2\ s^{-1}$), n (orbits per century), $n\Delta\varphi$: (arc sec/century), and GRT/LRGT-AF results. Earth mass $M_\oplus = 5.97 \times 10^{24}$ kg

Planet	Jupiter	Saturn	Uranus	Neptune
Mass	318	95.4	14.5	17.1
r_{min}(AU)	4.995	9.041	18.330	29.820
ϵ	0.0484	0.0539	0.0473	0.0095
Period	4333	10759	30687	60190
J	791	321	69.2	102
L_1	0.416	0.562	0.799	1.000
n	8.42	3.40	1.19	0.61
$n\Delta\varphi^*$	0.634×10^{-1}	0.137×10^{-1}	0.240×10^{-2}	0.777×10^{-3}
GRT/LRGT-AF**	0.621×10^{-1}	0.136×10^{-1}	0.237×10^{-2}	0.773×10^{-3}

TABLE III: The Solar System II. Mass planet m_{pl} in earth masses, $r_{min}(AU)$, ϵ eccentricity, $J = 2\pi m_{pl} r_{min}^2/T$: orbital angular momentum (units $10^{40} kg\ m^2/s$), L_1: angular momentum per unit mass (units $10^{16} m^2\ s^{-1}$), n (orbits per century), $n\Delta\varphi$: (arc sec/century), GRT/LRGT-AF results. Earth mass $M_\oplus = 5.97 \times 10^{24}$ kg and the solar mass $M_\odot = 1.98892 \times 10^{30}$ kg. Program calculation *), Literature **).

From the inspection of Schwinger's computation [32] Eqs. (2-4.36,37),

$$E_{int}(y^0) = -GM \int d^3y \, \frac{1}{|\mathbf{x}-\mathbf{y}|} \left[t^{00} - t_{kk}\right](y^0, \mathbf{y}),$$

and using (10.7) we get

$$\delta^{(1)} E_{int}(y^0) = -GM \int d^3y \, \frac{1}{|\mathbf{x}-\mathbf{y}|} \left[e^{-\mu_G |\mathbf{x}-\mathbf{y}|} - 1\right] t^{00}(y^0, \mathbf{y}), \tag{10.8a}$$

$$\delta^{(2)} E_{int}(y^0) = -\frac{1}{3} GM \int d^3y \, \frac{1}{|\mathbf{x}-\mathbf{y}|} \left[e^{-\mu_G |\mathbf{x}-\mathbf{y}|} - \cos(\mu_G |\mathbf{x}-\mathbf{y}|)\right] t^{00}(y^0, \mathbf{y}). \tag{10.8b}$$

With the Sun in the center $\mathbf{x} = 0$, using the limit $\mu_G < 10^{-57} m_e$ and for the distance Mercury-Sun $d := |\mathbf{r}_\odot - \mathbf{r}_m| \approx 60\ 10^{60} m = 1.44\ 10^{23} \hbar/m_e c$, we find that $\mu_G |\mathbf{y}| < 1.44\ 10^{-34} <<<< 1$.

Let $t^{00}(y^0, \mathbf{y}) \sim m/V_m$, i.e. a homogeous and static mass distribution inside Mercury. Then,

$$\begin{aligned}\delta E_{int}(y^0) &= \delta^{(1)}E_{int}(y^0) + \delta^{(2)}E_{int}(y^0) = GMm\Big[\left(e^{-\mu_G d} - 1\right) + \frac{1}{3}\left(e^{-\mu_G d} - \cos(\mu_G d)\right)\Big]/d \\ &\approx [GMm]\left[-\frac{4}{3} + \frac{5}{6}(\mu_G d)\right]\mu_G, \end{aligned} \tag{10.9}$$

where we expanded the exponential and cosine in (10.9) keeping only terms up to the quadratic ones in the graviton mass. The first term in (10.9) adds a constant to the potential energy, which gives no contribution to the gravitational force and hence no contribution to the perihelium-precession [13].

The second term: the equation of an orbit becomes

$$\frac{d^2u}{d\varphi^2} + u = -\frac{1}{L_1^2}\frac{d}{du}\frac{V_{eff}}{m},\ V_{eff}/m = V^{(0)}_{eff}/m + \frac{5}{3}GM\ \mu_G^2/u, \tag{10.10}$$

where $u = 1/R \equiv 1/d$ and L_1 is the angular momentum per unit planetary mass. From (10.9) we have

$$V_{eff}/m) = -GM\ u - 3G^2M^2u^2 + \frac{5}{3}GM\ \mu_G^2/u, \tag{10.11a}$$

$$\frac{d}{du}\frac{V_{eff}}{m} = -GM - 6G^2M^2u - \frac{5}{3}GM\left(\frac{\mu_G}{u}\right)^2, \tag{10.11b}$$

so that

$$\frac{d^2u}{d\varphi^2} + \left(1 - \frac{6G^2M^2}{L_1^2}\right)u = \frac{GM}{L_1^2} + \frac{5}{3L_1^2}\left(\frac{\mu_G}{u}\right)^2. \tag{10.12}$$

If u_0 is the orbit for $\mu_G = 0$ we have $1/u^2 \approx 1/u_0^2 - 2\Delta u/u_0^3$. Writing (10.12) as

$$\begin{aligned}&u'' + Au = B + C/u^2 \approx B + C\mu_G^2\left(\frac{1}{u_0^2} - 2\frac{u-u_0}{u_0^3}\right),\ \text{or}\\ &u'' + (A + 2C\mu_G^2/u_0^3)u \approx B + 3C\mu_G^2/u_0^2.\end{aligned}$$

and (10.12) can be written approximately as

$$\frac{d^2u}{d\varphi^2} + \left(1 - \frac{6G^2M^2}{L_1^2} + \frac{10GM}{3L_1^2}\frac{\mu_G^2}{u_0^3}\right)u = \frac{GM}{L_1^2}\left(1 + 5\left(\frac{\mu_G}{u_0}\right)^2\right). \tag{10.13}$$

The factor (...) on the r.h.s. means a constant multiple of the Newtonian potential $V = -GMm/R$ and does not produce a perihelium precession, it only changes slightly the scale of the orbit. The factor on the l.h.s. multiplying the u-term brings a scaling factor for the angle, and leads to a shift in the perihelium precession, quadratic in the graviton mass,

$$\delta\varphi = -2\pi\frac{5GM}{3L_1^2}\frac{\mu_G^2}{u_0^3}. \tag{10.14}$$

[13] Mass of the Sun $M_\odot = M = 1.99\ 10^{30}$ kg, mass of the Earth $M_\oplus = 5.97\ 10^{24}$ kg, mass of Mercury m= 0.053 $M_\oplus$, electron mass m_e= 9.11 10^{-31} kg.

The ratio with the $\Delta\varphi_E$ (10.6) is

$$\frac{\delta\varphi}{\Delta\varphi_E} = -\frac{5}{9}(\mu_G R)^2 \left(\frac{GM}{Rc^2}\right)^{-1} . \tag{10.15}$$

Now, $Gm_F^2/\hbar c$ is dimensionless, which implies that

$$\frac{GM}{Rc^2} = \left[G\frac{m_F^2}{\hbar c}\right] \frac{M}{m_F} \left[\frac{\hbar}{m_F c}/R\right]$$

is also dimensionless. Here $m_F = \hbar c = 197.32$ MeV is the Fermi mass. So, in atomic units $\hbar = c = 1$ we have

$$\frac{\delta\varphi}{\Delta\varphi_E} = -\frac{5}{9}\left(\frac{\mu_G}{m_F}\right)^2 (m_F R)^3 \frac{m_F}{M} \left[Gm_F^2\right]^{-1} . \tag{10.16}$$

The corrrection to Einstein's result is proportional to the square of the graviton mass and vanishes for $\mu_G \to 0$

Estimation: Using $\sqrt{G} = 1.62{\times}10^{-33}\ cm = 1.62{\times}10^{-20}\ fm$, one has $Gm_F^2 \approx 2.7{\times}10^{-40}$. The ratio $m_F/M \approx 400 m_e/M = 2{\times}10^{-58}$. Assuming $R \approx 10^8 km$, gives $m_F R \approx 10^{26}$, insertion these numbers in (10.16) gives

$$\frac{\delta\varphi}{\Delta\varphi_E} \approx -\frac{5}{9}\left(\frac{\mu_G}{m_F}\right)^2 \cdot 10^{+78} \cdot 2{\times}10^{-58} \cdot 10^{+40}/2.7 \approx -\frac{10}{27}\left(\frac{\mu_G}{m_F}\right)^2 \cdot 10^{60}. \tag{10.17}$$

From more recent work [9–12] the upper limit for the graviton mass seems to be $\mu_G \leq 7{\times}10^{-32}$ eV $= 2{\times}10^{-38}\ m_e \approx 0.5{\times}10^{-40}\ m_F$. This gives $|\delta\varphi/\Delta\varphi_E| \leq 10^{-21}$, which is very tiny.
The Einstein correction $\Delta\varphi_E = 43''.03$/century, and experiment gives $\Delta\varphi_{exp} = 41''.4 \pm 0''.90$/century [34]. For a deviation of the order of the error $\delta\varphi/\Delta\varphi_E \approx 0.01$ we find from (10.16) $\mu_G \approx 10^{-28} m_e$, i.e. ten orders of magnitude larger than the upper limit above.

B. Non-Newtonian Modified Gravity

The non-relativistic two-body gravitational potential is

$$V(r) = V_N(r) + V_{SC}(r) + V_{SG}(r) = -[GMm]\frac{e^{-\mu_G r}}{r} - \frac{GMm}{3r}\left[e^{-\mu_G r} - \cos(\mu_G r)\right] . \tag{10.18}$$

with $\mu_G \approx 10^{-40} m_F$. The range of the Yukawa part is $r_0 = 10^{25} m \approx 10^9$ light year. The correction to the Newton-potential in (10.18) is $\Delta V(r) = (2GMm/3c^2)\,[1 + \mu_G r/4 + ...]\,\mu_G c^2$, having a long-range repulsive part, which is very small. For $\mu_G d \approx 1 \to d \approx 2{\times}10^9$ ly, whereas the radius of the universe $r_U \approx 46.5 \times 10^9$ ly.

XI. DISCUSSION AND CONCLUSIONS

In studying the spin-2 theory with the axilary-field method, exploiting spin-1 and scalar spin-0 fields, we found analyzing the possible theories that only in the limiting case $b \to \infty$ a model with a massive spin-2 combined and imaginary-ghost spin-0 field has a proper massless limit. This gives a linearized relativistic gravitation theory in Minkowski space LRGT-AF. In the framework of LRGT-AF we performed the expansion in the (small) graviton mass, without destroying the correct prediction for the perihelium precession of Mercury. Therefore, in the treatment of the massive spin-2 field with auxiliary fields and the Dirac quantization method, a continuous and smooth change in the perihelium precession as a function of the graviton mass can be realized, albeit necessary to introduce a ghost-field. This ghost-field $\epsilon(x)$ plays a crucial role in (i) the field commutators, (ii) the Feynman propagator, and (iii) the correct massless limit. In this paper is the quantization of the imaginary-ghost performed, the Gupta-condition discussed for the Hilbert-space structure, the ghost-propagator analyzed, and the contribution to the perihelium precession calculated for the massive graviton. In the massless limit it converts the massive spin-projection operator into the massless one, which is very important for the LRGT-AF model.
The role of the ϵ-field is completely similar to that of the B-field in the massive vector-field [8]. In [8] section (2.4.2), the field equations are

$$
\begin{aligned}
&(\Box + m_V^2)A^\mu - (1-\alpha)\partial^\mu B = -ej^\mu, \\
&\partial^\mu A_\mu + \alpha B = 0,
\end{aligned}
$$

which are similar to equations (7.1). Also, the vector-field commutator has the vacuum expectation value

$$
\begin{aligned}
&\left(0\right| \left[A_\mu(x), A_\nu(y)\right] \left|0\right) = i m_V^{-2}\partial_\mu\partial_\nu \Delta(x-y;\alpha m_V^2) \\
&+i\int_0^\infty ds\rho(s)\left(-\eta_{\mu\nu} - s^{-1}\partial_\mu\partial_\nu\right)\,\Delta(x-y;s),
\end{aligned}
$$

which is the parallel of (7.5). Similarly for the Feynman propagator. So, the function of the ghost-fields $B(x)$ and $\epsilon(x)$ is very much the same. [14] They participate in the coupling to the electric current respectively to the matter-energy-momentum tensor.
The introduction of the asymptotic-fields reveals the physical contents of LRGT-AF as shown in the spectral analysis of the theory. It shows that in the $\lim M \to 0$ also the "renormalized" mass vanishes with the ratio $M_r/M \to 1$.
The eventual connection with the cosmological constant gives strong conditions on the possible mass μ_G of the graviton, leading to negligible corrections. Also, the non-newtonian corrections are very small for the solar-system.

The General-Relativity Theory (GRT) [26] has proven to be very succesful in describing an impressive amount of phenomena, particularly in Astrophysics and Cosmology. The search for deviations from GRT has been extensive but till now there is no sign of the presence of *e.g.* a Yukawa-like component in the graviton-exchange potential [12, 37]. *The study in this*

[14] A notable difference is the imaginary ϵ-mass. According to [6] an imaginary-mass field cannot be satisfactorily quantized. This may be a weak point of the model presented in this paper.

paper seems to confirm that it is impossible to disprove the conjecture by Van Dam and Veltman [1] within the context of the standard respected field theory models. The (sophisticated) spin-2 model with the imaginary-ghost is interesting, but its physical foundation is uncertain [6].

Appendix A: Quantization imaginary-mass field

We rescale the $\epsilon(x)$-field

$$\epsilon(x) := \frac{1}{2}\sqrt{3}\sqrt{\frac{b(1-b)^2}{(3+b)^3}\frac{M^2}{\mathcal{M}^2}}\,\widetilde{\epsilon}(x) \Rightarrow \frac{1}{2}\sqrt{3}\frac{M^2}{\mathcal{M}^2}\,\widetilde{\epsilon}(x), \tag{A1}$$

where $\Rightarrow$ indicates the limit $\lambda \to 1(b \to \pm\infty)$. The basic commutation relation (4.9a) for the scalar field becomes normalized

$$[\widetilde{\epsilon}(x), \widetilde{\epsilon}(y)] = -i\Delta(x-y; M_\epsilon^2), \tag{A2}$$

with $M_\epsilon = i\sqrt{2\lambda}M \Rightarrow i\sqrt{2}M$. The quantization of spinless complex-ghost fields has been discussed in e.g. [6, 22, 23]. In our discussion below we will follow these references. In particular we analyse the $\widetilde{\epsilon}(x)$ fields in terms of the spin-less complex-ghost fields $\phi(x)$ and $\phi^\dagger(x)$ having M_ϵ and M_ϵ^*, respectively. Compared to these references, the (-)-sign) in the commutator (A2) is different, and we will discuss the ensuing differences. Apart from the imaginary masses the situation is quite similar to that of the B-field in the so-called B-field formalism for (massive) vector field, cfr. [8], subsection 2.4.2. So, we introduce a Gupta subsidiary condition for the physical states $|f\rangle$ in the total Fock-space

$$\phi^{(+)}(x)|f\rangle = \phi^{\dagger(+)}(x)|f\rangle = 0, \tag{A3}$$

and the quantization procedure is quite analogous to that as described in [6, 22, 23] for the complex-ghost fields.

Notice that sofar we have not defined $\Delta(x-y; M_\epsilon^2)$. This is the topic of the rest of this section.

1. Imaginary-ghost Quantization

In order to obtain real potentials, we follow the quantization method given by Nakanishi [6] for the scalar field with an imaginary mass. We make the identification

$$\widetilde{\epsilon}(x) =: \frac{1}{\sqrt{2}}\left[\phi(x) + \phi^\dagger(x)\right]. \tag{A4}$$

Here, ϕ and $\phi^\dagger$ are spinless free complex-ghost fields having $\mu = +i\sqrt{2}\mu_G$ and $\mu^* = -i\sqrt{2}\mu_G$, respectively. The Lagrangian [6] is given by

$$\mathcal{L}_\phi = \frac{1}{2}\left(\partial^\alpha\phi\partial_\alpha\phi - \mu^2\phi^2 + \partial^\alpha\phi^\dagger\partial_\alpha\phi^\dagger - \mu^{*2}\phi^{\dagger 2}\right) \tag{A5}$$

The expansion of the field operator $\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x)$ in terms of annihilation and creation operators is, see [6], section 16,

$$\phi^{(+)}(x) = \int \frac{d^3p}{2\omega_p(2\pi)^3}\,\alpha(\mathbf{p})\exp\left(i\mathbf{p}\cdot\mathbf{x} - i\omega_p x_0\right), \tag{A6a}$$

$$\phi^{(-)}(x) = \int \frac{d^3p}{2\omega_p(2\pi)^3}\,\beta^\dagger(\mathbf{p})\exp\left(-i\mathbf{p}\cdot\mathbf{x} + i\omega_p x_0\right). \tag{A6b}$$

The canonical commutation relations for a ghost with a negative-metric imply [15]

$$
\begin{aligned}
,\left[\alpha(\mathbf{p}), \beta^\dagger(\mathbf{q})\right] &= \left[\beta(\mathbf{p}), \alpha^\dagger(\mathbf{q})\right] = -(2\pi)^3\delta(\mathbf{p}-\mathbf{q}), && \text{(A7a)}\\
\left[\alpha(\mathbf{p}), \alpha^\dagger(\mathbf{q})\right] &= \left[\beta(\mathbf{p}), \beta^\dagger(\mathbf{q})\right] = 0,\ \text{etc} && \text{(A7b)}
\end{aligned}
$$

from which follow the field-commutators

$$
\begin{aligned}
[\phi(x), \phi(y)] &= -i\Delta(x-y, i\mu), && \text{(A8a)}\\
\left[\phi(x), \phi^\dagger(y)\right] &= 0, && \text{(A8b)}\\
\left[\phi^\dagger(x), \phi^\dagger(y)\right] &= -i\Delta(x-y, -i\mu), && \text{(A8c)}
\end{aligned}
$$

with the two-point vacuum expectation values [6] eq. (16.32),

$$
\begin{aligned}
\langle 0|\phi(x)\phi(y)|0\rangle &= -\int \frac{d^3p}{2\omega_p(2\pi)^3}\exp\left[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y}) - i\omega_p(x_0-y_0)\right], && \text{(A9a)}\\
\langle 0|\phi(x)\phi^\dagger(y)|0\rangle &= 0, && \text{(A9b)}\\
\langle 0|\phi^\dagger(x)\phi^\dagger(y)|0\rangle &= -\int \frac{d^3p}{2\omega_p^*(2\pi)^3}\exp\left[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y}) - i\omega_p^*(x_0-y_0)\right]. && \text{(A9c)}
\end{aligned}
$$

where $\omega^* = -\omega$. This gives

$$
\begin{aligned}
\langle 0|[\phi(x), \phi(y)|0\rangle &= -\int \frac{d^3p}{(2\pi)^3\omega_p}\exp\left[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})\right]\sin\omega_p(x^0-y^0), && \text{(A10a)}\\
\langle 0|[\phi^\dagger(x), \phi^\dagger(y)|0\rangle &= -\int \frac{d^3p}{(2\pi)^3\omega_p^*}\exp\left[i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})\right]\sin\omega_p^*(x^0-y^0),. && \text{(A10b)}
\end{aligned}
$$

The field-commutator for the $\widetilde{\varepsilon}(x)$-field becomes

$$
[\widetilde{\varepsilon}(x), \widetilde{\varepsilon}(y)] = -\frac{i}{2}\Big(\Delta(x-y; i\mu) + \Delta(x-y; -i\mu)\Big). \tag{A11}
$$

2. Imaginary-ghost Propagator

Here, to emphasize the difference in mass, we used in the argument of the invariant Δ-function μ instead of μ^2. This implies for the Feynman-propagator of the $\widetilde{\varepsilon}(x)$-field [24]

$$
\begin{aligned}
i\Delta(x-y; M_\varepsilon^2) &\equiv \langle 0|T\left[\widetilde{\varepsilon}(x)\widetilde{\varepsilon}(y)\right]|0\rangle = \Big\{\langle 0|T\left[\phi(x)\phi(y)\right]|0\rangle + \langle 0|T\left[\phi^\dagger(x)\phi^\dagger(y)\right]|0\rangle\Big\}\\
&= -\frac{i}{2}\Big[\Delta_F(x-y; M=i\mu) + \Delta_F(x-y; M=-i\mu)\Big]. && \text{(A12)}
\end{aligned}
$$

The proper integration contour Γ, see Fig. 3, in the complex p_0-plane is given as [6] $\Gamma = R - \delta(\omega_p) + \delta(-\omega_p)$, where $\delta(\pm\omega_p)$ denotes a *counterclockwise circle* around the poles

[15] Note that, in contrast to the imaginary-mass case case treated in [6], section 16 and 17, we have here a ghost with a **negative metric**. This is taken care off by the (-)-sign on the r.h.s. in (A7a).

at $\pm\omega_p$ [25]. (Note that for a real mass M_ϵ the Γ-contour becomes the usual contour for the Feynman propagator C_F.)

Conjecture: The Feynman propagator function in (9.8) is

$$\widetilde{\Delta}(p;M_\epsilon^2) = \frac{1}{2}\left[\widetilde{\Delta}^{(1)}(p;M_\epsilon^2) + \widetilde{\Delta}^{(2)}(p;M_\epsilon^2)\right], \tag{A13a}$$

$$\widetilde{\Delta}^{(1)}(p;M_\epsilon^2) = \frac{1}{p^2 - M_\epsilon^2 + i0} + 2\pi i\,\delta(p^2 - M_\epsilon^2), \tag{A13b}$$

$$\widetilde{\Delta}^{(2)}(p;M_\epsilon^{*2}) = \frac{1}{p^2 - M_\epsilon^{*2} + i0}. \tag{A13c}$$

To show this we first evaluate for the integration contour $\Gamma = R - \delta(\omega_p) + \delta(-\omega_p)$ the contribution from the Leray coboundaries to $\widetilde{\Delta}^{(1)}(p;M_\epsilon^2)$ in the complex p_0-plane:

$$\delta(\omega_p) \sim -i\pi\, e^{i\mathbf{p}\cdot\mathbf{x} - i\omega_p x^0}/\omega_p, \quad \delta(-\omega_p) \sim +i\pi\, e^{i\mathbf{p}\cdot\mathbf{x} + i\omega_p x^0}/\omega_p,$$

and therefore

$$-\delta(\omega_p) + \delta(-\omega_p) = i\pi\left[e^{-i\omega_p x^0} + e^{+i\omega_p x^0}\right]\frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{\omega_p}. \tag{A14}$$

Next we evaluate

$$\begin{aligned} I_\delta \equiv& \int \frac{d^4p}{(2\pi)^4}\,\delta(p^2 - M^2) = \int \frac{d^3p}{(2\pi)^4}\, e^{i\mathbf{p}\cdot\mathbf{x}}\,. \\ &\times \int \frac{dp_0}{|2p_0|}\left[\delta(p^0 - \omega_p) + \delta(p^0 + \omega_p)\right] e^{-ip_0 x^0} \\ =& \int \frac{d^3p}{(2\pi)^4}\,\frac{1}{2\omega_p}\left[e^{-i\omega_p x^0} + e^{+i\omega_p x^0}\right] e^{i\mathbf{p}\cdot\mathbf{x}}. \end{aligned} \tag{A15}$$

Note: in the application here $M = i\mu$ and $\omega_p^2 = \mathbf{p}^2 - \mu^2$ is real. This justifies the use of $|p_0|$ in (A15). From the last two equations we conclude that

$$-\delta(\omega_p) + \delta(-\omega_p) \sim 2\pi i\,\delta(p^2 - M^2), \tag{A16}$$

which proofs the conjecture (QED).
For $M_\epsilon = i\mu$ and $M_\epsilon^* = -i\mu$ we have

$$\begin{aligned} \left[p^2 - M_\epsilon^2 + i0\right] &\to \mathcal{P}(p^2 + \mu^2)^{-1} - i\pi\delta(p^2 + \mu^2), \\ \left[p^2 - M_\epsilon^{*2} + i0\right] &\to \mathcal{P}(p^2 + \mu^2)^{-1} - i\pi\delta(p^2 + \mu^2). \end{aligned}$$

Therefore, using (A13), we find that with $M_\epsilon = -M_\epsilon^* = i\mu$

$$\widetilde{\Delta}(p;M_\epsilon^2) = \frac{1}{2}\left[\widetilde{\Delta}^{(1)}(p;M_\epsilon^2) + \widetilde{\Delta}^{(2)}(p;M_\epsilon^2)\right] = \mathcal{P}\frac{1}{p^2 + \mu^2}. \tag{A17}$$

We note that in the one-graviton-exchange diagram Fig. 2 we have on-energy-shell external particles that $p^0 = 0$. Then, $p^2 + \mu^2 \to -(\mathbf{p}^2 - \mu^2)$, which is used in (9.10).

Appendix B: Massless-limit Helicity-couplings

For massive gravitons the projection operator is [32]

$$\bar{\Pi}_{\mu\nu,\alpha\beta}(k) \;=\; \sum_{\lambda=1,5} \bar{\varepsilon}^{(\lambda)}_{\mu\nu}(k)\bar{\varepsilon}^{(\lambda)}_{\alpha\beta}(k) = \frac{1}{2}\left(\bar{\eta}_{\mu\alpha}\bar{\eta}_{\nu\beta} + \bar{\eta}_{\mu\beta}\bar{\eta}_{\nu\alpha}\right) - \frac{1}{3}\bar{\eta}_{\mu\nu}\bar{\eta}_{\alpha\beta}, \tag{B1}$$

with $\bar{\eta}_{\mu\nu} = \eta_{\mu\nu} - k^\mu k^\nu/M^2$.
The coupling to the conserved energy-momentum $t_{M,\mu\nu}$ warrants the replacement

$$\bar{\Pi}_{\mu\nu,\alpha\beta}(k) \;\to\; P^{(2)}_{\mu\nu;\alpha\beta} = \left\{\frac{1}{2}\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha}\right) - \frac{1}{3}\eta_{\mu\nu}\eta_{\alpha\beta}\right\} \tag{B2}$$

Next we show that in the massless limit, owing to the properties of the polarization vectors, the helicities $\lambda = -1, +1$ decouple smoothly from the $t_{M,\mu\nu}$-tensor for $\lim_{M\to 0}$, and that the helicity $\lambda = 0$ gives a contribution that is calcelled in the massless limit by a contribution from the scalar imaginary ghost.

1. Spin-2 Polarization Vectors

We work in a frame where $p^\mu = (p^0, 0, 0, p)$, for which we choose the spin-1 polarization vectors in the standard form:

$$\varepsilon^\mu(\pm 1) \;=\; \frac{1}{\sqrt{2}}(0, \mp 1, -i, 0) \;\;,\;\; \varepsilon^\mu(0) = \frac{1}{M}\left(p, 0, 0, E_p\right)\;, \tag{B3}$$

Then, the spin-2 polarization vectors are, up to a 'gauge' transformation,

$$\varepsilon^{\mu\nu}(p, \lambda = +2) \;=\; \varepsilon^\mu(p, +1)\,\varepsilon^\nu(p, +1)\,, \tag{B4a}$$

$$\varepsilon^{\mu\nu}(p, \lambda = +1) \;=\; \frac{1}{\sqrt{2}}\Big(\varepsilon^\mu(p, +1)\,\varepsilon^\nu(p, 0) + \varepsilon^\mu(p, 0)\,\varepsilon^\nu(p, +1)\Big)\;, \tag{B4b}$$

$$\varepsilon^{\mu\nu}(p, \lambda = 0) \;=\; \frac{1}{\sqrt{6}}\Big(2\varepsilon^\mu(p, 0)\,\varepsilon^\nu(p, 0) + \varepsilon^\mu(p, +1)\,\varepsilon^\nu(p, -1) + \varepsilon^\mu(p, -1)\,\varepsilon^\nu(p, +1)\Big) \tag{B4c}$$

and similarly for $\lambda = -1, -2$.

2. Polarization Vectors and Smooth Massless limit

We note that for small M the leading terms for the $\lambda = 0$ polarization are

$$\varepsilon^\mu(p, \lambda = 0) \;=\; \frac{p^\mu}{M} + \left(-\frac{M}{2p}, 0, 0, \frac{M}{2p}\right) + O(M^3) \tag{B5}$$

which gives

$$\varepsilon^{\mu\nu}(p, \lambda = +1) \;\sim\; \frac{1}{\sqrt{2}M}\Big(\varepsilon^\mu(p, +1)\, p^\nu + \varepsilon^\nu(p, +1)\, p^\mu\Big)\;, \tag{B6}$$

$$\varepsilon^{\mu\nu}(p, \lambda = 0) \;\sim\; \sqrt{\frac{2}{3}}\frac{p^\mu p^\nu}{M^2} + \frac{1}{\sqrt{6}}\left(\eta^{\mu\nu} - \frac{p^\mu \tilde{p}^\nu + \tilde{p}^\mu p^\nu}{2p\cdot\tilde{p}}\right)\;, \tag{B7}$$

where we introduced $\tilde{p}^\mu = (p^0, -\mathbf{p})$, and used the identity

$$\varepsilon^\mu(p,+1)\,\varepsilon^\nu(p,-1) + \varepsilon^\mu(p,-1)\,\varepsilon^\nu(p,+1) = \eta^{\mu\nu} - \frac{p^\mu \tilde{p}^\nu + \tilde{p}^\mu p^\nu}{2p\cdot\tilde{p}}\,. \tag{B8}$$

Terms in the polarization tensors $\varepsilon^{\mu\nu}(p,\lambda) \propto p^\mu, p^\nu$ *do not contribute to the matrix elements because* $p^\mu t_{M,\mu\nu} = 0$. *This also in the massless limit, since* $\lim_{M\downarrow 0}(p^\mu/M)t_{M,\mu\nu} = 0$. The only "false helicity", *i.e.* $\lambda \neq \pm 2$, that survives is $\varepsilon^{\mu\nu}(p,\lambda=0) \sim \eta^{\mu\nu}/\sqrt{6}$. which contributes a term $-\eta_{\mu\nu}\eta_{\alpha\beta}/6$ in $P^{(2)}_{\mu\nu;\alpha\beta}$. This term is cancelled in the massless limit by a contribution from the ϵ-ghost

$$-\frac{1}{3}\eta_{\mu\nu}\eta_{\alpha\beta}\widetilde{\Delta}_F(p^2,M^2) - \frac{1}{6}\eta_{\mu\nu}\eta_{\alpha\beta}\widetilde{\Delta}_F(p^2,M_\epsilon^2) = -\frac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta}\widetilde{\Delta}_F(p^2,M^2)$$
$$+\frac{1}{6}\eta_{\mu\nu}\eta_{\alpha\beta}\left[\widetilde{\Delta}_F(p^2,M^2) - \widetilde{\Delta}_F(p^2,M_\epsilon^2)\right],$$

where on the r.h.s. the second ("mass correction") term $\propto M_\epsilon^2 - M^2 \propto M^2$, showing the smooth transition to the massless case.

Appendix C: BS-equation and LS-equation

In this appendix we consider the Bethe-Salpeter equation (BSE) in the normalization used in [28] for scalar external particles. To start, we note that in [28] the Feynman rules give the invariant amplitude $-iM$. For scalar-exchange the potential is readily seen using the Feynman-rules [28] to be given as

$$V(P',p';P,p) = \frac{g^2}{q^2-\mu^2+i\delta} \tag{C1}$$

Here, the external momenta are (P',p') and (P,p) for the final and initial state respectively. The exchange momentum is $q = p'-p = P-P'$.

The total and relative momenta for the initial, final, and intermediate states are defined as

$$\begin{aligned} p_a &= \mu_a P + p,\;\; p_b = \mu_b P - p,\;\; p'_a = \mu_a P' + p',\;\; p'_b = \mu_b P' - p', \\ k_a &= \mu_a K_n + k,\;\; k_b = \mu_b K_n - k, K_n = k_a + k_b. \end{aligned} \tag{C2}$$

In the following we use for the weights $\mu_a = \mu_b = 1/2$. From the conservation of the total momenta, i.e. $P_i = P_f = K_n \equiv W$, the dependence of the amplitude and potential is given by

$$M(P',p';P,p) \equiv M(p_f,p_i;W),\;\; V(P',p';P,p) \equiv V(p_f,p_i;W). \tag{C3}$$

From an anlysis of the planar-box graph for scalar-exchange we infer the BSE, see Fig. 4, for scalar external particles as

$$M(P',p';P,p) \;=\; V(p_f,k_n;W) + \int d^4k_n\, V(p_f,k_n;W)\, G_n(k_n;W)\, M(k_n,p_i;W) \tag{C4}$$

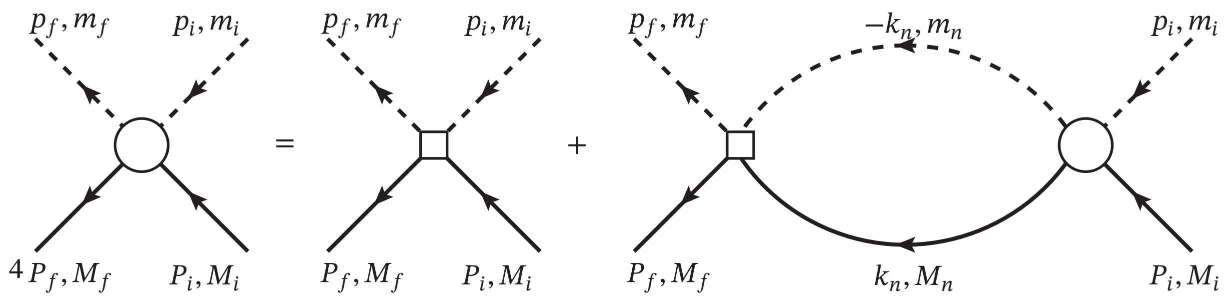

FIG. 4: *BS-Integral Equation*

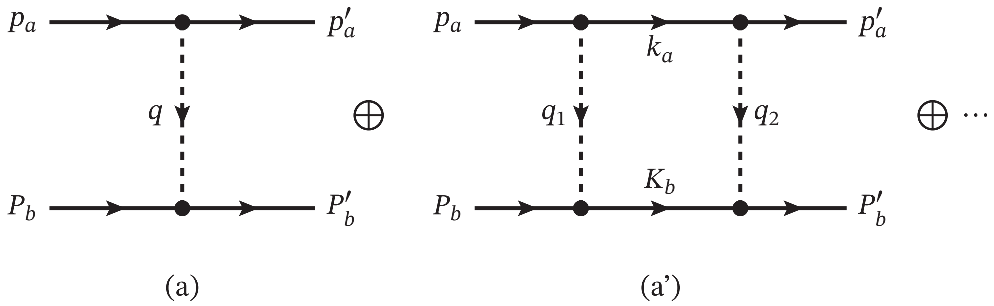

FIG. 5: One-meson and planar two-meson exchange etc. Feynman graphs. The solid lines denote scalar heavy particles, e.g., the sun and the planet. The dashed lines refer to the scalar mesons.

with

$$G_n(k_n; W) \;=\; \frac{i}{\left[(\frac{1}{2}W + k_n)^2 - M^2 + i\delta\right]\left[(\frac{1}{2}W - k_n)^2 - m^2 + i\delta\right]} \tag{C5}$$

Equation (C5) can easily be read off from the amplitude for the planar-box graph depicted in Fig. 5. In the application to planetary motion in these notes, the particles, planets and sun, off-energy-shell effects are non-existent. Therefore, the amplitude and potential are k_n^0-independent. The poles of the Green function $G_n(k_n; W)$, see Fig. 6, are at

$$\begin{aligned} \omega^{\pm}_{a_n} &= k^{0,\pm}_{n,a} = -\frac{1}{2}\sqrt{s} \pm \mathcal{E}(k_n) \mp i\delta, \\ \omega^{\pm}_{b_n} &= k^{0,\pm}_{n,b} = +\frac{1}{2}\sqrt{s} \pm E(k_n) \mp i\delta, \end{aligned} \tag{C6}$$

1. Positive and negative energy contributions: Integrating over k_n^0, using the residue theorem, in the r.h.s. of (C5) we get

$$\int_{-\infty}^{+\infty} dk_n^0\, G_n(k_n; W) = \pi \frac{\mathcal{E}(k_n) + E(k_n)}{\mathcal{E}(k_n)E(k_n)}\, \frac{1}{s - (\mathcal{E}(k_n) + E(k_n))^2}. \tag{C7}$$

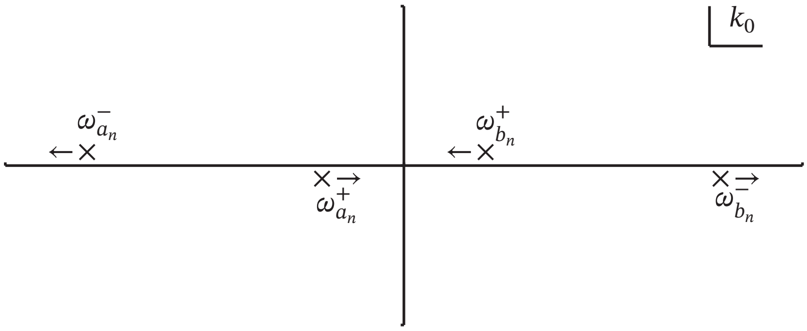

FIG. 6: Poles of the two-particle Green function.

In the low energy approximation, we have

$$s-(\mathcal{E}(k_n)+E(k_n))^2 \approx \frac{M+m}{m_{red}}\left(\mathbf{p}_i^2-\mathbf{k}_n^2\right), \quad \frac{\mathcal{E}+E}{\mathcal{E}\,E}(\mathbf{k}_n) \approx \frac{M+m}{Mm}\cdot$$
$$\times\left[1+\left(1-\frac{M^2+m^2}{Mm}\right)\frac{\mathbf{k}_n^2}{2Mm}\right] \sim \frac{1}{m}\left(1-\frac{\mathbf{k}_n^2}{2m^2}\right) \ (M \gg m). \tag{C8}$$

With this approximation the BSE (C4) becomes

$$M(p_f,p_i;W) \;=\; V(p_f,k_n;W)+\int d^4k_n\, V(p_f,k_n;W)\, g_n(k_n;W)\, M(k_n,p_i;W), \tag{C9a}$$
$$g_n(k_n;W) \;=\; \pi\frac{\mathcal{E}(k_n)+E(k_n)}{\mathcal{E}(k_n)E(k_n)}\,\frac{1}{s-(\mathcal{E}(k_n)+E(k_n))^2} \tag{C9b}$$
$$\approx\; \frac{\pi}{Mm(1+\mathbf{k}_n^2/2m^2}\,\frac{2m_{red}}{\mathbf{p}_i^2-\mathbf{k}_n^2+i\delta} \tag{C9c}$$

The transition to the Lippmann-Schwinger equation (LSE) is made by the transformation

$$\mathcal{T}(p_f,p_i)=N(p_f)\,M(p_f,p_i;W)\,N(p_i), \quad \mathcal{V}(p_f,p_i)=N(p_f)\,V(p_f,p_i;W)\,N(p_i), \tag{C10}$$

where

$$N(p)=\sqrt{2\pi\frac{\mathcal{E}(p)+E(p)}{\mathcal{E}(p)E(p)}} \approx \sqrt{\frac{\pi}{2Mm}}\left(1-\frac{\mathbf{p}^2}{4m^2}\right) \ (M \gg m). \tag{C11}$$

So the potential for the LSE at low energy becomes

$$\mathcal{V}(p_f,p_i) \;=\; N(p_f)\,V(p_f,p_i;W)\,N(p_i) \approx \frac{\pi}{2Mm}\left(1-\frac{\mathbf{p}_f^2+\mathbf{p}_i^2}{4m^2}\right)V(p_f,p_i;W)$$
$$=\; \frac{\pi}{2Mm}\left(1-\frac{\mathbf{q}^2+\mathbf{k}^2/4}{4m^2}\right)V(p_f,p_i;W). \tag{C12}$$

2. No negative energy contributions: We split the intermediate state propagators in the "positive" and "negative" part as follows:

$$\frac{1}{\left(\frac{1}{2}W+k_n\right)^2-M^2+i\delta}=\frac{1}{2\mathcal{E}(k_n)}\left[\frac{1}{\left(\frac{1}{2}\sqrt{s}+k_n^0\right)-\mathcal{E}+i\delta}-\frac{1}{\left(\frac{1}{2}\sqrt{s}+k_n^0\right)+\mathcal{E}-i\delta}\right],$$

$$\frac{1}{\left(\frac{1}{2}W-k_n\right)^2-m^2+i\delta}=\frac{1}{2E(k_n)}\left[\frac{1}{\left(\frac{1}{2}\sqrt{s}-k_n^0\right)-E+i\delta}-\frac{1}{\left(\frac{1}{2}\sqrt{s}-k_n^0\right)+E-i\delta}\right].$$

Neglecting the contributions of the negative-energy states the two-particle Green function is given by

$$G_n^{++}(k_n;W) = \frac{1}{2\mathcal{E}(k_n)E(k_n)}\left[\frac{1}{\frac{1}{2}\left(k_n^0+\sqrt{s}\right)-\mathcal{E}+i\delta}\cdot\frac{1}{\frac{1}{2}\left(\sqrt{s}-k_n^0\right)-E+i\delta}\right]. \quad \text{(C13)}$$

As above, the k_n^0-integration gives

$$\int_{-\infty}^{+\infty} dk_n^0\, G_n^{++}(k_n;W)=\frac{2\pi}{\mathcal{E}(k_n)E(k_n)}\,\frac{1}{\sqrt{s}-(\mathcal{E}(k_n)+E(k_n))}\equiv \mathrm{g}_n^{++}(k_n;W). \quad \text{(C14)}$$

In the low energy approximation, we obtain

$$\begin{aligned}\mathrm{g}_n^{++}(k_n;W) &= \frac{\pi}{2\mathcal{E}(k_n)E(k_n)}\,\frac{1}{\sqrt{s}-(\mathcal{E}(k_n)+E(k_n))}\\ &\approx \frac{\pi}{2Mm}\left(1+\frac{M^2+m^2}{2M^2m^2}\mathbf{k}_n^2\right)\frac{2m_{red}}{\mathbf{p}_i^2-\mathbf{k}_n^2+i\delta} \end{aligned} \quad \text{(C15)}$$

Again, the transition to the Lippmann-Schwinger equation (LSE) is made by the transformation

$$\mathcal{T}(p_f,p_i)=N(p_f)\,M(p_f,p_i;W)\,N(p_i),\quad \mathcal{V}(p_f,p_i)=N(p_f)\,V(p_f,p_i;W)\,N(p_i), \quad \text{(C16)}$$

where

$$N(p)=\sqrt{\pi/2\mathcal{E}(p)E(p)}\approx\sqrt{\frac{\pi}{2Mm}}\left(1-\frac{M^2+m^2}{4M^2m^2}\,\mathbf{p}^2\right)\ \ (M\gg m), \quad \text{(C17)}$$

and the potential for the LSE at low energy becomes

$$\begin{aligned}\mathcal{V}(p_f,p_i) &= N(p_f)\,V(p_f,p_i;W)\,N(p_i)\approx\frac{\pi}{2Mm}\cdot\\ &\times\left(1-\frac{M^2+m^2}{4m^2M^2}(\mathbf{q}^2+\mathbf{k}^2/4)\right)V(p_f,p_i;W). \end{aligned} \quad \text{(C18)}$$

Notice that for $M \gg m$ the form with no negative energy contribution (C18) is equivalent to (C12).

3. Non-local Potential and Schrödinger Equation: For a non-local potential, i.e.

$$\mathcal{V}(\mathbf{k},\mathbf{q})=\upsilon(\mathbf{k}\left(\mathbf{q}^2+\frac{1}{4}\mathbf{k}^2\right) \quad \text{(C19)}$$

the action on the wave function is [30]

$$\langle \mathbf{r}|\mathcal{V}|\psi\rangle = -\frac{1}{2}\left(\boldsymbol{\nabla}^2 v(\mathbf{r}) + v(\mathbf{r})\boldsymbol{\nabla}^2\right)\ \psi(\mathbf{r}). \tag{C20}$$

In [30] the ϕ-function is introduced as $\phi(r) = m_{red}\ v(r)$, and the radial Schrödinger equation, orbital angular momentum integer l, after making the Green-transformation $u_l = (1+2\phi)^{-1/2} w_l$,reads

$$w_l'' + \left[k^2 - 2m_{red}W(r) - l(l+1)/r^2\right]\ w_l(r) = 0. \tag{C21}$$

The "effective" potential W is energy dependent and given by

$$\begin{aligned} W(r) &= \frac{\mathcal{V}(r)}{1+2\phi} - \frac{1}{2m_{red}}\left(\frac{\phi'}{1+2\phi}\right)^2 + \frac{2\phi}{1+2\phi}\frac{p_i^2}{2m_{red}} \\ &\approx \mathcal{V}(r) - 2\phi(r)\left[\mathcal{V}(r) - \frac{p_i^2}{2m_{red}}\right] \end{aligned} \tag{C22}$$

Now, using a circular approximation the (classical) total energy is

$$E = \frac{p_i^2}{2m_{red}} = +\frac{1}{2}\mathcal{V} < 0\ \ (\mathbf{b.s.}) \tag{C23}$$

4. Perihelium-precession Planets: Here, we focus on the $1/r^2$-terms, which are responsible for the perihelium-precession. From the previous paragraph we have that

$$\phi(r) = \frac{7m_{red}}{4m^2}\ \mathcal{V}^{(0)}(r) \tag{C24}$$

There are now two possibilities:

a. We treat E in (C22) as a function of r as given in (C23), which gives

$$W^{(a)}(r) = \mathcal{V}(r) - \phi(r)\ \mathcal{V}(r), \tag{C25}$$

and consequently the $1/r^2$-correction is given by

$$\Delta\mathcal{V} \approx -\phi(r)\ \mathcal{V}^{(0)}(r) = -\frac{7}{4m}\left[\mathcal{V}^{(0)}\right]^2 \tag{C26}$$

which is $7/12\times$ Einstein's result.
b. We treat E in (C22) as a constant, like in [20, 32]. Then, the E-term in (C22) is of $1/r$-type and only deforms the shape of the orbit a little bit. The $1/r^2$-correction becomes

$$\Delta\mathcal{V} \approx -2\phi(r)\ \mathcal{V}^{(0)}(r) = -\frac{7}{2m}\left[\mathcal{V}^{(0)}\right]^2, \tag{C27}$$

which agrees with Schwinger [32], and leads to $7/6\times$ Einstein's result!

Note: *In a circular, classical, motion the kinetic and potential energy are connected by the equilibrium equation:*

$$|F_{grav.}| = |F_{centr.}| \to |\mathcal{V}'| = \frac{|\mathcal{V}|}{r} = \frac{mv^2}{r} = \frac{p^2}{mr}$$

or $T = |\mathcal{V}|/2$, *instead of* $T = E - \mathcal{V}$*!?*

Appendix D: Scalar-interaction Imaginary-ghost Field

The Yukawa-interaction of a scalar fields $\psi(x)$, $\chi(x)$ with the imaginary-ghost field $\phi(x)$ we describe by the interaction Hamiltonian

$$\mathcal{H}_I(x) = \frac{1}{2}\left[g_\psi\psi^2(x) + g_\chi\chi^2(x)\right]\,\left(\phi(x) + \phi^\dagger(x)\right), \tag{D1}$$

where the ψ and χ masses are M and m respectively. For the existence of the Dyson S-matrix a Gaussian adiabatic factor is necessary [6]

$$\mathcal{H}_I^\varepsilon(x_0) = \mathcal{H}_I\, e^{-\varepsilon x_0^2}, \tag{D2}$$

such that for the transition matrix $U_\varepsilon(x_0, y_0)$ the time limits $x_0 \to +\infty$ and $y_0 \to -\infty$ exist. Then, the S-matrix is given by

$$S_\varepsilon \;=\; \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int d^4x_1 \ldots \int d^4x_n\, T\left[\mathcal{H}_I^\varepsilon(x_1)\ldots\mathcal{H}_I^\varepsilon(x_n)\right]. \tag{D3}$$

The 2nd order S-matrix element is

$$\langle p', P'|S_\varepsilon^{(2)}|p, P\rangle \;=\; -\frac{1}{2}\int d^4x \int d^4y\, \langle p', P'|T\left[\mathcal{H}_I^\varepsilon(x)\mathcal{H}_I^\varepsilon(y)\right]|p, P\rangle. \tag{D4}$$

The one-particle states of the scalar particles give the wave functions

$$\psi_P(x) = \langle 0|\psi(x)|P\rangle = \left[(2\pi)^3 2\omega(P)\right]^{-1/2}\, e^{-iP\cdot x}, \tag{D5a}$$

$$\chi_p(x) = \langle 0|\chi(x)|p\rangle = \left[(2\pi)^3 2\omega(p)\right]^{-1/2}\, e^{-ip\cdot x}. \tag{D5b}$$

The plane wave expansion of the imaginary-ghost field is [6]

$$\phi(x) \;=\; \int \frac{d^3p}{\sqrt{(2\pi)^3 2\omega_p}}\left[\alpha(\mathbf{p})e^{-ip\cdot x} + \beta^\dagger(\mathbf{p})e^{+ip\cdot x}\right],$$

$$\phi^\dagger(x) \;=\; \int \frac{d^3p}{\sqrt{(2\pi)^3 2\widetilde{\omega}_p}}\left[\alpha^\dagger(\mathbf{p})e^{-ip\cdot x} + \beta(\mathbf{p})e^{-ip\cdot x}\right]. \tag{D6a}$$

The quantization, such that $[\phi(x), \phi^\dagger(y)] = 0$ and the negative-metric, is given by the commutation relations

$$\left[\alpha(\mathbf{p}), \beta^\dagger(\mathbf{q})\right] = \left[\beta(\mathbf{p}), \alpha^\dagger(\mathbf{q})\right] = -(2\pi)^3\delta(\mathbf{p} - \mathbf{q}), \tag{D7a}$$

$$\left[\alpha(\mathbf{p}), \alpha^\dagger(\mathbf{q})\right] = \left[\beta(\mathbf{p}), \beta^\dagger(\mathbf{q})\right] = 0. \tag{D7b}$$

For imaginary-ghost exchange between the scalar particles the relevant term in the Wick-expansion of $T[\ldots]$ is given by

$$T\left[\psi^2(x)\phi(x)\,\chi^2(y)\phi(y)\right] \Rightarrow N\left[\psi^2(x)\,\chi^2(y)\right]\langle 0|T\left[\phi(x)\,\phi(y)\right]|0\rangle, \tag{D8}$$

where

$$\langle 0|T\,[\phi(x)\,\phi(y)]\,|0\rangle = i\Delta_F(x-y;i\mu), \tag{D9}$$

and

$$\begin{aligned}\langle p',P'|N\left[\psi^2(x)\,\chi^2(y)\right]|p,P\rangle \;=\; & (2\pi)^{-6}\left[16\omega_{p'}\omega_{P'}\omega_p\omega_P\right]^{-1/2}\cdot \\ & \times\left\{e^{i(p'-p)\cdot x}e^{i(P'-P)\cdot y}+e^{i(p'-p)\cdot y}e^{i(P'-P)\cdot x}\right\}.\end{aligned} \tag{D10}$$

The 2nd order S-matrix element becomes

$$\begin{aligned}\langle p',P'|S_\varepsilon^{(2)}|p,P\rangle \;=\; & -g_\psi g_\chi\,(2\pi)^{-3}\left[16\omega_{p'}\omega_{P'}\omega_p\omega_P\right]^{-1/2}\int d^4x\int d^4y\;\cdot \\ & \times e^{i(p'-p)\cdot x}e^{i(P'-P)\cdot y}\;i\Delta_F(x-y;i\mu)\,e^{-\varepsilon x_0^2}e^{-\varepsilon y_0^2}.\end{aligned} \tag{D11}$$

Using now the variables

$$Z=\frac{1}{2}(x+y),\;\; z=x-y, \tag{D12}$$

and defining the M-matrix by

$$S_\varepsilon(f,i)=\delta_{f,i}-(2\pi)^4 i\,\delta(P_f-P_i)\,M_\varepsilon(f,i), \tag{D13}$$

we obtain

$$\begin{aligned}\langle p',P'|M_\varepsilon^{(2)}|p,P\rangle \;=\; & g_\psi g_\chi\,(2\pi)^{-3}\left[16\omega_{p'}\omega_{P'}\omega_p\omega_P\right]^{-1/2}\cdot \\ & \times\int d^4z\;e^{i(p'-p)\cdot z}\,\Delta_F(z;i\mu)\,e^{-\varepsilon z_0^2/2}\end{aligned} \tag{D14}$$

A similar contribution to $M_\varepsilon^{(2)}$ comes from the exchange of a $\phi^\dagger$ imaginary-ghost particle. In that case the Feynman propagator is

$$i\Delta_F(z;-i\mu)=\theta(z_0)\,\langle 0|]\phi^\dagger(z)\phi^\dagger(0)|0\rangle+\theta(-z_0)\,\langle 0|]\phi^\dagger(0)\phi^\dagger(z)|0\rangle. \tag{D15}$$

Working out the z_0-integrals in (D14) for $\Delta_F(z;-i\mu)$ explicitly, we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}dz_0\;e^{-\varepsilon z_0^2/2}\theta(z_0)\,\langle 0|]\phi^\dagger(z)\phi^\dagger(0)|0\rangle \Rightarrow \\ &-\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}d\tau\;e^{-\varepsilon\tau^2/2}\theta(\tau)\,e^{-i\widetilde{\omega}_p\tau}=i\left[\mathcal{P}\frac{1}{\widetilde{\omega}_p}+i\pi\delta(\widetilde{\omega}_p)\right],\end{aligned} \tag{D16}$$

and

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}dz_0\;e^{-\varepsilon z_0^2/2}\theta(-z_0)\,\langle 0|]\phi^\dagger(0)\phi^\dagger(z)|0\rangle \Rightarrow \\ &-\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}d\tau\;e^{-\varepsilon\tau^2/2}\theta(-\tau)\,e^{+i\widetilde{\omega}_p\tau}=i\left[\mathcal{P}\frac{1}{\widetilde{\omega}_p}+i\pi\delta(\widetilde{\omega}_p)\right],\end{aligned} \tag{D17}$$

and we get

$$\int_{-\infty}^{\infty}dz_0\;\Delta_F(z;-i\mu)=2\pi\int\frac{d^3p}{2\widetilde{\omega}_p(2\pi)^3}\,e^{i\mathbf{p}\cdot\mathbf{z}}\left[\frac{1}{\pi}\mathcal{P}\frac{1}{\widetilde{\omega}_p}+i\delta(\widetilde{\omega}_p)\right]. \tag{D18}$$

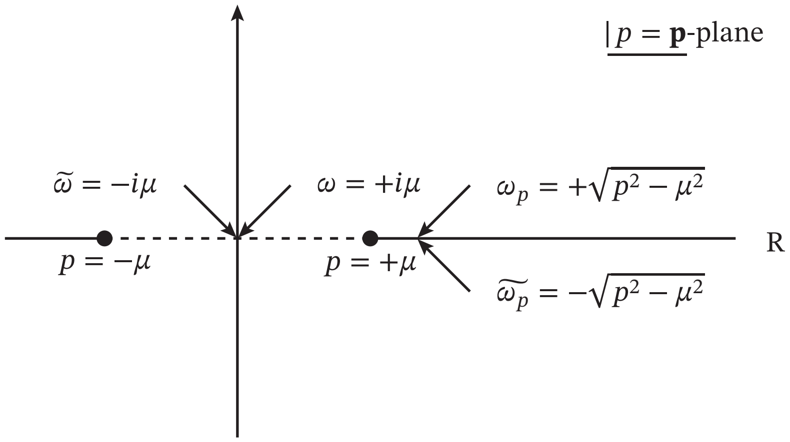

FIG. 7: Branchpoints $\omega(p)$ at $p = \pm\mu$, and the relation between $\omega(p)$ and $\widetilde{\omega}(p)$. The dashed-line indicates the branchline. On the upper rim of the branchline $\omega = i\sqrt{\mu^2 - p^2}$, and $\widetilde{\omega} = -i\sqrt{\mu^2 - p^2}$. For $\mu < p < \infty$ one has $\omega = \sqrt{p^2 - \mu^2}$, whereas $\widetilde{\omega} = -\sqrt{p^2 - \mu^2}$. With these choices of the branches of the square-root $\widetilde{\omega} = -\omega$.

Similarly for $\Delta_F(z; +i\mu)$

$$\int_{-\infty}^{\infty} dz_0 \, \Delta_F(z; i\mu) = 2\pi \int \frac{d^3 p}{2\omega_p (2\pi)^3} \, e^{i\mathbf{p}\cdot\mathbf{z}} \left[\frac{1}{\pi} \mathcal{P} \frac{1}{\omega_p} + i\delta(\omega_p) \right]. \tag{D19}$$

Here, for the branches of ω_p and $\widetilde{\omega}_p$ see Fig. 7,

$$\omega_p = \begin{cases} +\sqrt{\mathbf{p}^2 - \mu^2} \; (\mathbf{p}^2 > \mu^2), \\ +i\sqrt{\mu^2 - \mathbf{p}^2} \; (\mathbf{p}^2 < \mu^2) \end{cases} \tag{D20a}$$

$$\widetilde{\omega}_p = \begin{cases} -\sqrt{\mu^2 - \mathbf{p}^2} \; (\mathbf{p}^2 > \mu^2), \\ -i\sqrt{\mu^2 - \mathbf{p}^2} \; (\mathbf{p}^2 < \mu^2) \end{cases} \tag{D20b}$$

For the further evaluation of (D19) and (D18) we consider

$$\begin{aligned} J &\equiv \frac{1}{2\pi r} \int_0^\infty \frac{p dp}{\omega_p} \sin(pr) \left[i\delta(\omega_p) + \frac{1}{\pi} \mathcal{P} \frac{1}{\omega_p} \right] \\ &= \frac{1}{2\pi r} \left[-i \int_0^\mu \frac{p dp}{\sqrt{\mu^2 - p^2}} + \int_\mu^\infty \frac{p dp}{\sqrt{p^2 - \mu^2}} \right] \sin(pr) \cdot \\ &\quad \times \left[i\delta(\omega_p) + \frac{1}{\pi} \mathcal{P} \frac{1}{\omega_p} \right] \equiv \frac{1}{2\pi r} (K_1 + K_2) \,. \end{aligned} \tag{D21}$$

Using the identity [16]

[16] According to Eq. (16.44) of [6] the Dirac δ-function and Cauchy's Principal value for complex argument, being

$$\delta(\omega_p) = \delta\left(\sqrt{\mu^2 - p^2}\right) = \frac{1}{p}\sqrt{\mu^2 - p^2}\,[\delta(p-\mu) + \delta(p+\mu)]\,, \tag{D22}$$

we obtain for K_1 and K_2

$$\begin{aligned} K_1^\varepsilon &= +\frac{1}{2}\sin(\mu r) + \frac{1}{\pi}\int_0^{\mu-\varepsilon} \frac{pdp}{p^2-\mu^2}\sin(pr), \\ K_2^\varepsilon &= +\frac{i}{2}\sin(\mu r) + \frac{1}{\pi}\int_{\mu+\varepsilon}^{\infty} \frac{pdp}{p^2-\mu^2}\sin(pr), \end{aligned} \tag{D23}$$

Here, a factor 1/2 in included in the δ-term because of the endpoint situation. Next we evalute the similar integrals for the $\phi^\dagger$-propagation.

$$\begin{aligned} \widetilde{J} &\equiv \frac{1}{2\pi r}\int_0^\infty \frac{pdp}{\widetilde{\omega}_p}\,\sin(pr)\left[i\delta(\widetilde{\omega}_p) + \frac{1}{\pi}\mathcal{P}\frac{1}{\widetilde{\omega}_p}\right] \\ &= \frac{1}{2\pi r}\left[+i\int_0^\mu \frac{pdp}{\sqrt{\mu^2-p^2}} - \int_\mu^\infty \frac{pdp}{\sqrt{p^2-\mu^2}}\right]\sin(pr)\cdot \\ &\quad \times\left[i\delta(\widetilde{\omega}_p) + \frac{1}{\pi}\mathcal{P}\frac{1}{\widetilde{\omega}_p}\right] \equiv \frac{1}{2\pi r}\left(\widetilde{K}_1 + \widetilde{K}_2\right). \end{aligned} \tag{D24}$$

For $\widetilde{K}_1$ and $\widetilde{K}_2$ we obtain

$$\begin{aligned} \widetilde{K}_1^\varepsilon &= -\frac{1}{2}\sin(\mu r) + \frac{1}{\pi}\int_0^{\mu-\varepsilon} \frac{pdp}{p^2-\mu^2}\sin(pr), \\ \widetilde{K}_2^\varepsilon &= -\frac{i}{2}\sin(\mu r) + \frac{1}{\pi}\int_{\mu+\varepsilon}^{\infty} \frac{pdp}{p^2-\mu^2}\sin(pr), \end{aligned} \tag{D25}$$

From these results we find

$$\frac{1}{2}\left(J + \widetilde{J}\right) = +\frac{1}{\pi}\mathcal{P}\int_0^\infty \frac{pdp}{p^2-\mu^2}\sin(pr). \tag{D26}$$

The time-integral over the propagator for the field $\widetilde{\varepsilon} = (\phi + \phi^\dagger)/\sqrt{2}$ becomes

$$\begin{aligned} &\int_{-\infty}^{\infty} dz_0\,\frac{1}{2}\left[\Delta_F(z; i\mu) + \Delta_F(z; -i\mu)\right] = \\ &\frac{1}{2\pi^2 r}\mathcal{P}\int_0^\infty \frac{qdq}{q^2-\mu^2}\sin(qr) = \mathcal{P}\int \frac{d^3q}{(2\pi)^3}\,\frac{\exp(i\mathbf{q}\cdot\mathbf{r})}{\mathbf{q}^2-\mu^2}. \end{aligned} \tag{D27}$$

---

useful in extracting the finite parts of the Dyson S-matrix, can be defined by

$$\begin{aligned} \delta(\omega) &= \lim_{\varepsilon\to 0}\delta_\varepsilon(\omega)\ ,\ \ \delta_\varepsilon(\omega) = \int_{-\infty}^{\infty}\frac{d\tau}{2\pi}e^{-i\omega\tau}e^{-\frac{1}{2}\varepsilon\tau^2}, \\ P\left(\frac{1}{\omega}\right) &= \lim_{\varepsilon\to 0}P_\varepsilon\left(\frac{1}{\omega}\right)\ ,\ \ P_\varepsilon\left(\frac{1}{\omega}\right) = \frac{1}{2}\int_{-\infty}^{\infty} d\tau\,\sigma(\tau)\,e^{-i\omega\tau}e^{-\frac{1}{2}\varepsilon\tau^2}, \end{aligned}$$

From this definition it can be verified that $\delta(ix) = \delta(x)$.

and we obtain

$$
\begin{aligned}
\langle p', P'|M_\varepsilon^{(2)}|p, P\rangle &= g_\psi g_\chi\,(2\pi)^{-3}\left[16\omega_{p'}\omega_{P'}\omega_p\omega_P\right]^{-1/2}\cdot \\
&\quad\times\int d^3r\, e^{-i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{r}}\cdot\mathcal{P}\int\frac{d^3q}{(2\pi)^3}\,\frac{\exp(i\mathbf{q}\cdot\mathbf{r})}{\mathbf{q}^2-\mu^2} \\
&= g_\psi g_\chi\,(2\pi)^{-3}\left[16\omega_{p'}\omega_{P'}\omega_p\omega_P\right]^{-1/2}\mathcal{P}\frac{1}{(\mathbf{p}'-\mathbf{p})^2-\mu^2},
\end{aligned}
\tag{D28}
$$

which justifies the use of the principal-value integral in Eq. (9.10) etc.

Appendix E: Perihelium precession: -1/6-correction

The order G^2 contributions to the perihelium precession evaluated in sections VIII, IX and X, differ from the Einstein result by a factor 7/6. Here, we evaluate in detail an additional effect of order G^2 in the interaction between m and M. This is associated with the gravitational energy between the planet (mas m) and the Sun (mass M), which is not localized on either mass [32]. It is distributed in space and can be calculated from the Newtonian field strength:

$$
\mathbf{g}(\mathbf{x};\mathbf{r}) = G\,\boldsymbol{\nabla}\left[\frac{M}{|\mathbf{x}|}+\frac{m}{|\mathbf{x}-\mathbf{r}|}\right]. \tag{E1}
$$

	Newtonian Gravity	Electrostatics
Force between two sources	$\mathbf{F}_N = -\frac{GmM}{r^2}\hat{\mathbf{r}}$	$\mathbf{F}_C = +\frac{qQ}{r^2}\hat{\mathbf{r}}$
Force derived from potential	$\mathbf{F}_N = -m\boldsymbol{\nabla}\Phi_N(\mathbf{x}_M)$	$\mathbf{F}_C = -q\boldsymbol{\nabla}\Phi_C(\mathbf{x}_M)$
Potential outside spherical source	$\Phi_N = -\frac{GM}{r},\ \mathbf{g} = -\boldsymbol{\nabla}\Phi_N(\mathbf{x}_M$	$\Phi_C = \frac{Q}{r},\ \mathbf{E} = -\boldsymbol{\nabla}\Phi_C(\mathbf{x}_M$
Field equation for potential	$\boldsymbol{\nabla}^2\Phi_N = 4\pi G\mu(\mathbf{x}_M)$	$\boldsymbol{\nabla}^2\Phi_C = -\rho_{elec}(\mathbf{x}_M)$

TABLE IV: Newtonian Gravity and Electrostatics. The positions and charges of the masses M and m are $\mathbf{x}_M$ and $\mathbf{x}_m$, respectively Q and q. The relative distance is $\mathbf{r} = \mathbf{x}_m - \mathbf{x}_M$.

The energy density in a Newtonian gravitational field can be derived as follows: Consider the assembling of a system of N particles of mass M_A at the positions $\mathbf{x}_A$. The Newtonian potential energy W of the system is the needed energy by bringing them one by one from infinity in the gravitational field of all particles already assembled, which is

$$
W(\mathbf{r}) = -\frac{1}{2}\sum_{A\neq B}\frac{GM_AM_B}{|\mathbf{x}_A-\mathbf{x}_B|}. \tag{E2}
$$

For a continuous distribution of mass with density $\mu(\mathbf{x})$ this is

$$
W(\mathbf{r}) = -\frac{1}{2}\int d^3x\int d^3x'\,\frac{G\mu(\mathbf{x})\mu(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{2}\int d^3x\,\mu(\mathbf{x})\,\Phi_N(\mathbf{x}). \tag{E3}
$$

Via the Newtonian field equation $\boldsymbol{\nabla}^2\Phi_N(\mathbf{x}) = 4\pi G\ \mu(\mathbf{x})$ one can eliminate the source $\mu(\mathbf{x})$ in the last expression of Eq. (E3) and applying the divergence theorem leads to

$$W(\mathbf{r}) = -\frac{1}{8\pi G}\int d^3x\ \boldsymbol{\nabla}\Phi_N(\mathbf{x})\cdot\boldsymbol{\nabla}\Phi_N(\mathbf{x}) = -\frac{1}{8\pi G}\int d^3x\ [\mathbf{g}(\mathbf{x})]^2 \equiv \int d^3x\ \rho_{grav}(\mathbf{x}). \qquad \text{(E4)}$$

Here $\rho_{grav}(\mathbf{x})$ is the energy density associated with the Newtonian gravitational field.
The gravitational energy density in the gravitational field of the masses M and m is

$$\rho_{grav}(\mathbf{x};\mathbf{r}) = -\frac{1}{8\pi}G\ \boldsymbol{\nabla}\left(\frac{M}{|\mathbf{x}|}+\frac{m}{|\mathbf{x}-\mathbf{r}|}\right)\cdot\boldsymbol{\nabla}\left(\frac{M}{|\mathbf{x}|}+\frac{m}{|\mathbf{x}-\mathbf{r}|}\right). \qquad \text{(E5)}$$

Here, the gravitational self-energy terms, proportional to M^2 and m^2, are in principle incorporated into M and m and independent of to each others presence. (Moreover, these self-energy terms are independent of the positions and hence can not contribute to the perihelium precession.) The cross term, which contains the correlation of the two bodies, is $\mathbf{r}$-dependent and given by

$$\rho_{cross}(\mathbf{x};\mathbf{r}) = -\frac{1}{4\pi}G\ \boldsymbol{\nabla}\left(\frac{M}{|\mathbf{x}|}\right)\cdot\boldsymbol{\nabla}\left(\frac{m}{|\mathbf{x}-\mathbf{r}|}\right). \qquad \text{(E6)}$$

Because of the mass-energy equivalence, the energy density $\rho_{cross}(\mathbf{x};\mathbf{r})$ implies also a mass distribution $\mu_{cross} = \rho_{cross}/c^2$, and hence gives a gravitational pull to the planet and sun. The interaction energy of this energy density with M is of order G^2 and given by (units c=1)

$$\begin{aligned}\mathcal{V}_{cross,M}(\mathbf{r}) &\equiv -\int d^3x\ \frac{MG}{|\mathbf{x}|}\ \mu_{cross}(\mathbf{x};\mathbf{r}) = \frac{1}{4\pi}G^2M^2m\int d^3x\left(\frac{1}{|\mathbf{x}|}\boldsymbol{\nabla}\frac{1}{|\mathbf{x}|}\right)\cdot\boldsymbol{\nabla}\frac{1}{|\mathbf{x}-\mathbf{r}|}\\ &= \frac{1}{8\pi}G^2M^2m\int d^3x\ \boldsymbol{\nabla}\frac{1}{|\mathbf{x}|^2}\cdot\boldsymbol{\nabla}\frac{1}{|\mathbf{x}-\mathbf{r}|} = -\frac{1}{8\pi}G^2M^2m\ \cdot\\ &\quad\times\int d^3x\ \frac{1}{|\mathbf{x}|^2}\cdot\boldsymbol{\nabla}^2\frac{1}{|\mathbf{x}-\mathbf{r}|} = +\frac{1}{2}G^2M^2m\ \frac{1}{|\mathbf{r}|^2} = +\frac{V^2}{2m}.\end{aligned} \qquad \text{(E7)}$$

Here, we used $|\mathbf{x}|^{-1}\boldsymbol{\nabla}|\mathbf{x}|^{-1} = \boldsymbol{\nabla}|\mathbf{x}|^{-2}/2$, applied partial integration, $\boldsymbol{\nabla}^2(1/r) = -4\pi\delta(\mathbf{r})$, and the Newtonian potential $V \equiv -GMm/r$. In total one has for the perihelium precession $-7V^2/2m + V^2/2m = -6V^2/2m$, which is Einstein's result.

Note: In this note we give a detailed derivation which is symmetric between the planet and the sun. Similar to the mass M, the interaction energy of the energy density ρ_{cross} with the planet, mass m, is given by

$$\begin{aligned}\mathcal{V}_{cross,m}(\mathbf{r}) \equiv & -\int d^3x\,\frac{mG}{|\mathbf{x}|}\,\mu_{cross}(\mathbf{x};\mathbf{r}) = \frac{1}{4\pi}G^2Mm^2\cdot\int d^3x\,\left(\frac{1}{|\mathbf{x}-\mathbf{r}|}\boldsymbol{\nabla}\frac{1}{|\mathbf{x}|}\right)\cdot \\ & \cdot\boldsymbol{\nabla}\frac{1}{|\mathbf{x}-\mathbf{r}|} = +\frac{1}{2}G^2Mm^2\,\frac{1}{|\mathbf{r}|^2} = +\frac{V^2}{2M} \ll \mathcal{V}_{cross,M}(\mathbf{r}) \text{ for } m \ll M.\end{aligned} \tag{E8}$$

The total potential energy due to $\mu_{cross}(\mathbf{x},\mathbf{r})$ is

$$\begin{aligned}\mathcal{V}_{cross}(\mathbf{r}) &= \mathcal{V}_{cross,M}(\mathbf{r}) + \mathcal{V}_{cross,m}(\mathbf{r}) \\ &= \frac{1}{2}G^2Mm\,(M+m)\frac{1}{|\mathbf{x}_A-\mathbf{x}_B|^2} = \frac{M+m}{2Mm}\,V^2,\end{aligned} \tag{E9}$$

where $\mathbf{x}_A$ and $\mathbf{x}_B$ are the position of the sun and the planet respectively, $\mathbf{r} = \mathbf{x}_B - \mathbf{x}_A$, and $V = -GMm/r$. *The potential energies $\mathcal{V}_{cross,M}$ and $\mathcal{V}_{cross,m}$ are both sensitive to the position of the sun and the planet, and therefore leads to a force between the planet and the sun.* Notice the symmetry w.r.t. $M \leftrightarrow m$ which ensures the < *action = -reaction* > rule.
In the center-of-mass the separation of the relative motion, taking $M_A = M$, $M_B = m$, is as follows:

$$M\frac{d^2\mathbf{x}_A}{dt^2} = +\boldsymbol{\nabla}_A\mathcal{V}_{cross} = +\frac{1}{2}\frac{M+m}{Mm}\,\boldsymbol{\nabla}_r V^2, \tag{E10a}$$

$$m\frac{d^2\mathbf{x}_B}{dt^2} = +\boldsymbol{\nabla}_B\mathcal{V}_{cross} = -\frac{1}{2}\frac{M+m}{Mm}\,\boldsymbol{\nabla}_r V^2, \tag{E10b}$$

This gives for the center of mass $d^2\mathbf{R}_{c.m.}/dt^2 = 0$, and for the relative motion

$$\frac{d^2}{dt^2}(\mathbf{x}_B-\mathbf{x}_A) = -\frac{1}{2}\left(\frac{1}{M}+\frac{1}{m}\right)\frac{M+m}{Mm}\,\boldsymbol{\nabla}V^2 = -\frac{1}{2}\left(\frac{M+m}{Mm}\right)^2\boldsymbol{\nabla}V^2 \Rightarrow \tag{E11a}$$

$$\mu_{red}\frac{d^2\mathbf{r}}{dt^2} = -\boldsymbol{\nabla}\left(\frac{V^2}{2\mu_{red}}\right) \approx -\boldsymbol{\nabla}\frac{V^2}{2m}. \tag{E11b}$$

Here, $\mu_{red} = Mm/(M+m) \approx m$ for $m \ll M$.

Application of (E11) to the planet-sun system demonstrates that the potential in Eq. (E7) indeed represents to a very good approximation the proper extra potential from the ρ_{cross} energy distribution.

Appendix F: Cosmological constant and Graviton mass

The gravitational action including the cosmological term reads [38, 39]

$$S_g = -\frac{c^3}{16\pi G}\int d^4x\,\sqrt{-g}\,R - \frac{1}{c}\int d^4x\,\sqrt{-g}\,\Lambda. \tag{F1}$$

The Einstein equation follows from

$$\frac{\delta S_g}{\delta g_{\mu\nu}} = -\frac{c^3}{16\pi G}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)\sqrt{-g} + \frac{\Lambda}{2c}\, g^{\mu\nu}\sqrt{-g} = 0, \tag{F2}$$

or, with the inclusion of the matter term $\delta S_M/\delta g_{\mu\nu}$,

$$\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right) + \lambda\, g^{\mu\nu} = -\kappa T^{\mu\nu}/c^2, \tag{F3}$$

with Einstein's constant $\kappa = 8\pi G/c^2$ and $\lambda = 8\pi G\Lambda/c^4 = \kappa\Lambda/c^2$. Here is used, see [40], $\delta\sqrt{-g}/\delta g_{\mu\nu} = -g^{\mu\nu}\sqrt{-g}/2$. Eqn. (F3) is Einstein's equation Ref. [41], see also Ref. [42] equation (8.1.39).

Incorporation of the cosmological term in the spin-2 formalism of this paper, in the weak-field approximation, is achieved by the change $\mathcal{L}^{(2)} \to \mathcal{L}^{(2)} + c_0\sqrt{-g}$ in the spin-2 Lagrangian, with $c_0 = \Lambda/c$. In the weak field approximation this becomes , see Appendix G,

$$\mathcal{L}^{(2)} \;\to\; \mathcal{L}^{(2)} + c_0\left[1 + \frac{1}{2}\kappa h^\mu_\mu - \kappa^2\left(\frac{1}{4}h^\mu_\nu h^\nu_\mu - \frac{1}{8}(h^\mu_\mu)^2\right)\right] \tag{F4}$$

With the constraint $h^\mu_\mu = 0$, coming from $\partial\mathcal{L}_{\eta\epsilon}/\partial\epsilon(x) = 0$, only the $c_0\, h^{\mu\nu}h_{\mu\nu}/4$-term is relevant. This implies that in the Klein-Gordon equation (2.10) for the $h^{\mu\nu}$-field $M_2^2 \to M_2^2 + c_0\kappa^2$. Assuming that the origin of the gravitational mass is entirely due to the cosmological constant we have $\mu_G = M_2 = \sqrt{\Lambda/c}\,\kappa$.
The Friedmann equation reads [43], see also [42] eqn. (9.73) with $\Lambda = \lambda$,

$$H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{R^2} + \frac{1}{3}\lambda c^2. \tag{F5}$$

Note that for $R \gg 1$ the density becomes

$$\rho = \frac{3H^2}{8\pi G} - \frac{\lambda c^2}{8\pi G} = \rho_c - \frac{1}{16\pi G}\left(\frac{\mu_G c^2}{\hbar}\right)^2. \tag{F6}$$

The sign of the λ-term is in agreement with [43] Eqn. (9.1), but is opposite to that in Ref. [36] Eqn. (10.27) which has $\Lambda \Rightarrow -\mu_G^2 c^2/\hbar^2$, and implies the presence of "dark matter". $\Lambda < 0$ leads to an Anti-deSitter space for an empty universe, which seems unphysical. At the present epoch the Hubble constant is

$$H_0^2 = \frac{8\pi G}{3}\rho_0 - \frac{kc^2}{R_0^2} + \frac{1}{3}\lambda c^2. \tag{F7}$$

The deceleration parameter $q_0 = -\ddot{R}/RH^2$ satisfies [44]

$$q_0 \equiv -\ddot{R}/RH^2 = \frac{1}{2}\Omega_0 - c^2\Lambda/3H_0^2. \tag{F8}$$

From observations the deceleration parameter $|q_0| < 5$ [47], which gives

$$|\Lambda| \leq 21 H_0^2/c^2 \approx 10^{-54}\ cm^{-2}, \tag{F9}$$

for $H_0 \leq 100$ km s^{-1} Mpc^{-1}. With $M_{Pl} = 10^{-33}$ cm^{-1}, one has $|\Lambda|/M_{Pl}^2 < 10^{-120}$.

In the GUT picture, before the breakdown of the GUT gauge-symmetry via a first-order phase transition at the critical temperature $T_c \approx 10^{14}$ GeV, the (GUT) cosmological constant is much larger than the present one and is given by

$$\Lambda \equiv \frac{8\pi G}{3} T_c^4 \approx (10^{15} GeV)^4 M_{Pl}^{-2}. \tag{F10}$$

Interpretation of the cosmological constant term as a mass term in the equation of the $h^{\mu\nu}$-field we have [36]

$$\mu_G = \sqrt{2\Lambda} = \sqrt{\frac{16\pi G}{3}}\, T_c^2 = \sqrt{\frac{16\pi}{3}} \left(\frac{T_c}{M_{Pl}}\right)^2 M_{Pl} = 4.1\ 10^{-10} M_{Pl}, \tag{F11}$$

The perihelium precession of Mercury imposes a limit on the present cosmological constant, which follows from the modification of the Schwarzschild metric, namely

$$ds^2 = c^2 \left(1 - 2M/r - \frac{1}{3}\Lambda r^2\right) dt^2 - \left(1 - 2M/r - \frac{1}{3}\Lambda r^2\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \tag{F12}$$

where $M = M_{\odot} G/c^2$. From the accuracy of the value of the perihelium precession of Mercury one derives that, see [44]

$$|\Lambda| < 10^{-42} cm^{-2} = 10^{-108}\, M_{Pl}^2 \rightarrow \mu_G = \sqrt{2\Lambda} = 1.4\ 10^{-54} M_{Pl} \approx 2.8\ 10^{-32}\ m_e, \tag{F13}$$

where is used $M_{Pl}^{-1} = 10^{-33}$ cm, in units $\hbar = c = 1$.
The upper limit from the LIGO-Virgo Collaboration is $\mu_G < 1.76\times 10^{-23} eV/c^2 = 3.44\times 10^{-29} m_e$ [45].
The solar system upper limit of the graviton mass is $m_g^{SS} < 4.4 \times 10^{-22} eV/c^2$ giving $\lambda_g^{SS} > 2.8 \times 10^{12}$ km. LISA measurements with an "ideal source" could give an improvement by a factor ~ 50 [46].
The transition between the large cosmological constant Λ in (F10) and the tiny one in (F13) can be understood within the inflational phase transition scenario [48]. For this the "latent heat" ΛM_{pl}^2 is during this phase transition transformed into radiation, diminishing enormously the cosmological constant.

Appendix G: Miscellaneous formulas

For a diagonalizable matrix A

$$\det(I + \varepsilon A) \ = \ 1 + \varepsilon\, f_1(A) + \varepsilon^2\, f_2(A) + ... \tag{G1}$$

The first order term is $f_1(A) = Tr(A)$. To calulate the second order, we use

$$I + \varepsilon A \ = \ \exp(\varepsilon B),\ \det(\varepsilon B) = \exp[\varepsilon Tr(B)] \tag{G2}$$

This leads to

$$\begin{aligned}\det(I+\varepsilon A) &= \det[\exp(\varepsilon B)] = \det\left(I+\varepsilon B+\frac{\varepsilon^2}{2}B^2+\ldots\right)\\ &= 1+\varepsilon\, Tr(B)+\frac{\varepsilon^2}{2}\left(Tr\ B\right)^2+\ldots. \end{aligned} \tag{G3}$$

Also we can rewrite

$$\begin{aligned}\det(I+\varepsilon A) &= \det[\exp(\varepsilon B)] = \det\left[I+\varepsilon\left(B+\frac{\varepsilon}{2}B^2+\ldots\right)\right]\\ &= 1+\varepsilon\, Tr\left(B+\frac{\varepsilon}{2}B^2+\ldots\right)+\varepsilon^2 f_2\big(B+\ldots\big)+O(\varepsilon^3). \end{aligned} \tag{G4}$$

Taking $\lim \varepsilon \to 0$ leads to

$$f_2(B) = \frac{1}{2}\big[\left(Tr\ B\right)^2 - Tr\ \left(B^2\right)\big]. \tag{G5}$$

Since in this limit A=B we have $f_2(A) = f_2(B)$.
For $g_{\mu\nu} = \eta_{\mu\nu} + \kappa\ h_{\mu\nu}$ or in term of matrices $g = \eta + \kappa\ h$ we introduce

$$I = D\,\eta\,D\ ,\ A = D\,h\,D\ ,\ \varepsilon = \kappa, \tag{G6}$$

where the diagonal matrix D has $D_{00} = 1, D_{mm} = i\ (m = 1, 2, 3)$. Then,

$$\det(I+\varepsilon\ A) = \det\big(D\,(\eta+\kappa\ h)\,D\big) = -\det\big(\eta+\kappa\ h\big), \tag{G7}$$

since $\det\ D = -i$. Using the result above we obtain

$$\det\big(\eta+\kappa\ h\big) = -\det(I+\varepsilon\ A) = -\big(1+\varepsilon\ f_1(A)+\varepsilon^2 f_2(A)+\ldots\big) \tag{G8}$$

This gives

$$\begin{aligned}\sqrt{-\det(g)} &= \big[-\det\big(\eta+\kappa\ h\big)\big]\\ &= \big(1+\varepsilon\ f_1(A)+\varepsilon^2 f_2(A)+\ldots\big)^{1/2}\\ &= 1+\frac{1}{2}\varepsilon\ f_1(A)+\frac{1}{2}\varepsilon^2 f_2(A)-\frac{1}{8}\varepsilon^2 f_1^2(A)+O(\varepsilon^3) \end{aligned} \tag{G9}$$

Now, $Tr\ A = Tr(DhD) = Tr(D^2h) = Tr(\eta h)$ which gives

$$\sqrt{-\det(g)} = 1+\frac{1}{2}\varepsilon\ Tr(\eta h)+\frac{1}{2}\varepsilon^2\left[\frac{1}{2}(Tr(\eta h))^2-\frac{1}{2}Tr((\eta h)^2)-\frac{1}{4}(Tr(\eta h))^2\right]+\ldots \tag{G10}$$

Up to the second order in the gravitation constant κ we obtained

$$\sqrt{-\det(g)} = 1+\frac{1}{2}\kappa\ h^{\mu}_{\mu}-\kappa^2\left[\frac{1}{4}h^{\mu\nu}h_{\nu\mu}-\frac{1}{8}h^{\mu}_{\nu}h^{\nu}_{\nu}\right]. \tag{G11}$$

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$$\begin{aligned} \text{positive} - \text{metric} &: \langle 0|T\left[\phi(x)\phi(y)\right]|0\rangle = +i\Delta_F(x-y), \\ \text{negative} - \text{metric} &: \langle 0|T\left[\phi(x)\phi(y)\right]|0\rangle = -i\Delta_F(x-y). \end{aligned}$$

This is obvious from the derivation of the vacuum expectation values of $\langle 0|\phi(x)\phi(y)|0\rangle$ given in [28], section (12.4).
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$$\left|\frac{\boldsymbol{\nabla} V^{(0)} \cdot \boldsymbol{\nabla}\psi}{Mm\, V^{(0)}\psi}\right| \sim [Mm\, R_m^2]^{-1} \ll 1,$$

justifying the neglect of the $\boldsymbol{\nabla}\psi$ term in the planetary motion.

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$$detA = \sum_{j=1}^{n} a_{ij} A_{ij},\ A_{ij} = (-)^{i+j}\Delta_{ij}, \tag{G12}$$

where Δ_{ij} is the minor w.r.t. the ij-element of A, i.e. the determinant of the matrix obtained from A by deleting row i and column j. The minor is related to the inverse of A

$$(A^{-1})_{ij} = (detA)^{-1} A_{ji} \tag{G13}$$

which implies that for each row i

$$detA = \sum_{j=1}^{n} a_{ij} \cdot (A^{-1})_{ji}(detA), \tag{G14}$$

which implies

$$\frac{\partial detA}{\partial a_{ij}} = (A^{-1})_{ji}(detA). \tag{G15}$$

Because

$$g_{\mu\nu} g^{\nu\lambda} = \delta_\mu^\lambda,\ g^{\mu\nu} = g^{\nu\mu}, \tag{G16}$$

application of (G15) leads to

$$\delta\sqrt{-g}/\delta g_{\mu\nu} = -\frac{1}{2} g^{\mu\nu}\sqrt{-g}. \tag{G17}$$

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