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# The Optimal Polynomial Theory applied to 0–350 MeV *pp* scattering*

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## Abstract

The Optimal Polynomial Theory (OPT) of Cutkosky, Deo, and Ciulli has been tested for its use in a multi-energy phase shift analysis of *pp* scattering data below $T_{\text{lab}} = 350$ MeV. The power of OPT to predict higher partial wave phase parameters is investigated also for a realistic potential model, in casu the Nijmegen potential. It is seen that OPT has indeed predictive power whenever the phase parameters do not decrease too rapidly as a function of the orbital angular momentum. For a high-quality phase shift analysis OPT does not predict $F$- and $G$-waves well enough. Therefore these have to be parametrized. The predictive power of OPT is then only used for higher partial waves. It appears that the Nijmegen potential tail contains valuable physical information beyond OPT.

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## I. INTRODUCTION

The conventional partial wave (p.w.) expansion expresses the nuclear scattering amplitude as an infinite series of Legendre polynomials $P_\ell(\theta)$, where the expansion coefficients are related to the p.w. phase parameters. Due to the short range of the nuclear forces the higher p.w. phase parameters decrease quite rapidly. Usually one approximates therefore in a nucleon-nucleon phase shift analysis [1,2] the higher p.w. nuclear phase parameters, which are not searched for, by the longest range interaction, one-pion-exchange (OPE). One of the problems of an analysis is to judge which phase parameters can be determined from the data, and which phase parameters can be taken to be OPE. The quality of an analysis can be improved if one has a better approximation for the higher p.w. phase parameters.

The Optimal Polynomial Theory (OPT) of Cutkosky and Deo [3] and Ciulli [4] is meant to be a method to expand a function in a series in the most efficient way. The basic idea of OPT is to use knowledge about the domain of analyticity of the function to be expanded. For the scattering amplitude this means that the expansion is not made in terms of $P_\ell(\theta)$, but in functions which are determined by the branchpoint that is nearest to the physical region. The p.w. series is known to converge in the $\cos\theta$-plane within an ellipse, determined by the mass of the lightest exchanged particle (the Lehmann ellipse [5]). When applied in a *pp* phase shift analysis, the advantage of using OPT instead of the conventional p.w. expansion would be that one could reach the same accuracy with less parameters, since the OPT expansion converges faster. It provides also a smoother transition from parametrized to fixed phase parameters.

Cutkosky and Deo applied OPT to determine coupling constants from single-energy *np* and $K^+p$ differential cross sections [6]. The OPT method has first been tested for *pp* scattering by Chao [7] to investigate whether the higher p.w. phase parameters of the analyses of MacGregor *et al.* [8] from 200 to 400 MeV could be reproduced with OPT from their lower p.w. phase parameters. It later turned out [9] that for the triplet phase parameters the method followed by Chao did not work in practice. Rijken *et al.* [10–12] studied the use of a different set of triplet amplitudes, and found that the higher p.w. phase parameters of the Nijmegen soft core (NSC) potential [13] could be predicted by OPT from the lower ones for a number of partial waves. An acceleration of convergence of the p.w. series, more primitive than OPT, has been used in the single-energy phase shift analyses from 142 to 800 MeV of Dubois *et al.* [14].

In this paper we test the use of OPT for the description of the *pp* scattering data with $T_{\rm lab} \leq 350$ MeV.

We do not make use of the refinements in data analysis as developed by Cutkosky [15]. These were used by Chao [7] in his application of OPT to the *pp* and *np* phase shift analysis. We only check first whether the applied OPT formalism [11,12] is capable to produce the higher partial waves from the lower ones for the Nijmegen potential model. Secondly, we apply our OPT formalism to a *pp* phase shift analysis and compare the results with those obtained from using the Nijmegen potential tail for the higher partial waves.

The outline of the paper is as follows. In the next section the method is explained using a simple example. The specifics of OPT when applied to *pp* scattering are given in Section III. Here we note that we apply OPT to the $K$-matrix and not to the $T$-matrix. For the application to the $T$-matrix we refer to Cutkosky and Deo [6] and Chao [7]. In

Section IV the predictive power of OPT is judged from a realistic nuclear potential model. The predicted higher p.w. phase parameters can in that case be compared with the higher p.w. phase parameters of the model. In Section V, where OPT is used in a multi-energy *pp* phase shift analysis, we compare the quality of description of the data, measured by $\chi^2_{\rm min}$/d.o.f., with and without using OPT. In Section VI the results are summarized and discussed. Furthermore, an appendix is devoted to the detailed construction of the optimal mappings.

## II. THE METHOD: AN EXAMPLE

To illustrate the method we take a simple example. Let us assume that we are interested in an expansion on the interval $[-1,1]$ of the function $f(x) = (x_c - x)^{-1/2}$ with $x_c = 1.2$. This function has branchpoints at $x = x_c$ and $x = \infty$, and the cut can be taken along $[x_c, \infty)$.

For an expansion in a finite number of powers of $x$ we have to choose a norm in order to define which approximation is best. Using the $L_2$ inner product $\int dx\, fg$ on $[-1,1]$ results in the approximation

$$f^{(N)}(x) = \sum_{n=0}^{N} a_n P_n(x) \quad . \tag{1}$$

The expansion coefficients $a_n$ are independent of $N$ because the $P_n(x)$ are orthogonal with respect to the chosen inner product.

To make an OPT expansion of the function, one first has to construct a conformal mapping $z(x)$ which maps the complex $x$-plane on to an ellipse in the $z$-plane (Fig. 1).

The upper and lower edge of the cut are mapped on to the border of the ellipse; the points $-1$ and $1$ are invariant. This makes the mapping unique. In Appendix A we give the detailed construction of this mapping. We can now expand our example function in Legendre polynomials of $z(x)$. With the $L_2$ inner product $\int dz\, fg$ on $[-1,1]$ we obtain the approximation

$$f^{(N)}_{OPT}(x) = \sum_{n=0}^{N} c_n P_n(z(x)) \quad . \tag{2}$$

In Fig. 2 we show the result for a two term expansion. We also show the two term expansion in $x$ that would result without OPT. The difference should be clear from the figure.

To see why the OPT expansion should be better than the normal one, we look again at Fig. 1. It is known that a Legendre expansion converges within the largest unifocal (foci $\pm 1$) ellipse that contains no singularity. Now in the $z$-plane this ellipse is larger than in the $x$-plane (see Fig. 1). Convergence in a larger ellipse automatically means that the expansion coefficients decrease faster. This hopefully will lead to a more accurate approximation on $[-1,1]$ with a given number of terms. Unfortunately, one can only be sure of this if the number of terms in the expansion becomes sufficiently large. With a given number of terms, the quality of approximation depends strongly on the function that is approximated. For instance, the figure analogous to Fig. 2 for $f(x) = \ln(x_c - x)$ would have been even more impressive, but the results for $f(x) = \sqrt{x_c - x}$ would have been meager with a two term expansion.

Knowing only the position of the nearest branchpoint, OPT is the best expansion scheme to choose. Ciulli [4] gave a detailed mathematical treatment of the method. The expansion scheme of OPT is 'optimal', only if the expansion coefficients are determined using the function on the entire interval $[-1, 1]$. This is in fact what a Legendre expansion does. In practice, the available data do not form a continuum on this interval, but merely some spots on it. The distribution along $[-1, 1]$ only exists in a statistical sense. Of course, the situation of sparse data favors an expansion with not too many terms, which OPT seems to provide.

The OPT expansion (Eq. 2) in a finite number of powers of $z$ can be rewritten as an infinite series in $x$

$$f^{(N)}_{OPT}(x) = \sum_{n=0}^{\infty} \widetilde{a}_n P_n(x) \quad . \tag{3}$$

In Table I the coefficients $\widetilde{a}_n$ for a two term expansion (labeled OPT) are compared with the exact values. The coefficient $\widetilde{a}_2$ comes out fairly well. The higher coefficients are in the right direction, but too small in this example. Measured with the $L_2$ norm in $x$, the OPT expansion with two parameters is more than 2 times as good as the normal expansion with two parameters. Therefore, the use of OPT corrects for a substantial part of the truncation error that would be present in a two term expansion in $x$. It is seen from the table that $\widetilde{a}_0 \neq a_0$ and $\widetilde{a}_1 \neq a_1$. If one knows the lower expansion parameters (as in this example), one can choose a different method of determining the OPT expansion parameters, *e.g.* fixing $c_0$ and $c_1$ such that $\widetilde{a}_0 = a_0$ and $\widetilde{a}_1 = a_1$. The coefficients $\widetilde{a}_n$ of this method, labeled OPT′, are also given in Table I. In Fig. 2 the differences between both OPT approximations would be small. The OPT′ method is somewhat better near $x = -1$ and somewhat worse near $x = 1$.

The method analogous to OPT′ will be used in Section IV, where the OPT higher p.w. phase parameters are compared with model values. In the actual analysis the lower p.w. phase parameters are parametrized as smooth functions of the energy. The OPT expansion coefficients that reproduce these lower p.w. phases are determined by the criterium that they give the minimal $\chi^2$ in a fit to the scattering data.

It is clear that the coefficients $\widetilde{a}_n$ follow from the OPT coefficients $c_n$. For the first $N$ coefficients $\widetilde{a}_n$ there will be a one to one correspondence to the $c_n$. So instead of saying that OPT uses the $c_n$ as parameters to approximate $f(x)$, one can also argue that the first $N$ of the coefficients $\widetilde{a}_n$ are the parameters. In that case, OPT is seen as a kind of expansion in $x$, where the higher coefficients $\widetilde{a}_n$ with $n > N$ are 'predicted'. We will use this point of view, which is completely equivalent to the former, in our application to *pp* scattering. The reason for this is of course the connection with the existing expansion in $x = \cos\theta$, where the coefficients are related to the phase parameters.

Since $P_0(x) = P_0(z) = 1$, it is readily seen from Eqs. 2 and 3 that the $\widetilde{a}_0$ term (which in the phase shift analysis will correspond to the $^1S_0$ and $^3P_0$ phase shifts) is only linked to the $c_0$. Therefore the $\widetilde{a}_0$ term has no predictive power.

## III. THE METHOD: SPECIFICS FOR $PP$ SCATTERING

In this section we turn to the specifics of OPT when applied in a *pp* scattering phase shift analysis. First of all, we use the nuclear scattering amplitude, *i.e.* the scattering amplitude minus the electromagnetic amplitude, as the function to be expanded. The 'original' variable is $x = \cos\theta$, with $\theta$ the c.m. scattering angle. The procedure is rather simple for the spin-singlet amplitude. Usually this amplitude is expanded as

$$M_S(x) = \sum_{\ell=0}^{\infty}(2\ell+1)\alpha_\ell P_\ell(x) \quad , \tag{4}$$

where the sum is over even orbital angular momentum $\ell$ only. Thus $M_S$ is an even function. Its singularity structure, which is also symmetric, is given in Fig. 3. It only has cuts along $(-\infty, -x_c]$ and $[x_c, \infty)$, with

$$x_c = 1 + \frac{\mu^2}{2k^2} \ , \tag{5}$$

where $k$ is the c.m. relative momentum and $\mu$ is the mass of the lightest particle of which the exchange is taken into account in $M_S$. For our application of OPT as explained above, we have therefore $\mu = m_{\pi^0}$. For uncharged identical particles OPE leads to poles in the amplitude at $x = \pm x_c$. Due to the Coulomb interaction OPE leads for *pp* scattering to branchpoints at these places, as the amplitude then contains the factor $1/(q^2+\mu^2)^{1+2i\eta}$ [16], with $q$ the exchanged momentum and $\eta$ the Coulomb parameter.

A small difference between the method followed by Cutkosky and Deo and ours is that Cutkosky and Deo let OPT predict the higher $\alpha_\ell$ or $S-1$ matrix elements followed by a unitarization, whereas we predict the higher $K$-matrix elements. Since these $K$-matrix elements are real, unitarity is automatically guaranteed in the higher partial waves.

We can now proceed in two different ways. The first one is the method of Cutkosky and Deo [3]. It uses a mapping of this twice cut complex $x$-plane on to an ellipse in the $z$-plane. This is again a unique mapping, but it is different from the one we used for one cut. The construction of this mapping is again deferred to Appendix A. We only mention here that the OPT variable $z(x)$ will be an odd function of $x$. This means that we will need only even powers to expand $M_S$ in $z$. The other method has been proposed by Rijken *et al.* [10–12]. It was intended to be used for amplitudes, which in general have a right- and left-hand cut, although not always symmetrical. It starts with a decomposition of the amplitude

$$M(x) = M_L(x) + M_R(x) \ , \tag{6}$$

where $M_L$ and $M_R$ have only the left-hand and the right-hand cut respectively. This decomposition can be made uniquely if $M(x)$ behaves well at $x \to \infty$. In the general case, one then defines the signature amplitudes

$$M_\pm(x) = \frac{1}{2}\left(M_R(x) \pm M_L(-x)\right) \ . \tag{7}$$

The amplitudes $M_+$ and $M_-$ are then expanded separately. In our case, scattering of identical particles, the amplitudes are either symmetric or antisymmetric. Therefore one of the signature amplitudes vanishes, and the other coincides with $M_R$. In the case of our singlet amplitude we have for symmetry reasons

$$M_S(x) = M_L(x) + M_R(x) = M_R(-x) + M_R(x) \ . \tag{8}$$

Therefore we now have to expand only one function with only a right-hand cut. So we need here the optimal mapping that we first introduced. We will call this mapping the signature mapping. A disadvantage of this approach is that $M_R(x)$ is no longer an even function, so both even and odd powers appear in the expansion, but an advantage is that the optimal mapping for one cut results in a larger Lehmann ellipse than that for two cuts, so the expansion coefficients decrease somewhat faster.

We now turn to the problem of the spin-triplet amplitudes. In that case it is not quite clear which functions to expand. We cannot take the $M$-matrix elements $M_{00}$, $M_{01}$, $M_{10}$, $M_{11}$, and $M_{1-1}$ [17] as they are not independent and some of them have kinematical singularities at $x = \pm 1$. Chao [7] made use of amplitudes based on Fermi-invariants, which have been studied by Goldberger *et al.* [18]. It later turned out [9] that for the triplet amplitudes this procedure does not work. Another method is to make use of the so-called 'derivative' amplitudes of De Alfaro *et al.* [19]. Their use in $NN$ scattering has been studied by Rijken *et al.* [10–12]. With these derivative amplitudes it is possible to construct a set of functions that are closely related to the phase parameters and still have the required analyticity properties. They are

$$\begin{aligned}
\omega_1(x) &= \sum_J (2J+1)\alpha_{JJ} P_J(x) \\
\omega_{11}(x) &= \sum_J (2J+1)\alpha_{J-1,J} P_J(x) \\
\omega_{22}(x) &= \sum_J (2J+1)\alpha_{J+1,J} P_J(x) \\
\omega_{12}(x) &= \sum_J (2J+1)\alpha^J P_J(x) \quad ,
\end{aligned} \tag{9}$$

where we use the SYM parametrization [20] to define the p.w. amplitudes $\alpha_{\ell J}$. The functions $\omega_i$ are either odd or even in $x$, as shown in Table II. Furthermore the table lists the phase parameters involved in each $\omega_i$. For easier recognition of the p.w. phase parameters involved, the amplitudes $\omega_i$ are in the following called after the lowest p.w. phase parameter that they contain. The amplitudes are then called partial wave sequences. The OPT expansion for these functions can of course be made with the Cutkosky and Deo as well as with the signature mapping. For $\omega_1$, which is an odd function of $x$, the Cutkosky and Deo mapping uses only odd powers of $z(x)$, and the signature mapping has $\omega_{1L} = -\omega_{1R}$.

Finally we mention a method which makes explicit use of the OPE contribution $\omega_{1\pi}$ to the amplitudes. If we write

$$\omega(x) = \omega_{1\pi}(x) + \tilde{\omega}(x) \ , \tag{10}$$

then $\tilde{\omega}$ will have a more distant branchpoint (see Fig. 3). More explicitly we have $\mu = 2m_{\pi^0}$ in Eq. 5. We can now use OPT to expand the functions $\tilde{\omega}_i$ (also for the spin-singlet of course). This new method we call OPT($\pi$). If OPT is viewed as a method to predict higher p.w. phase parameters, then OPT($\pi$) predicts the deviation of higher p.w. phase parameters from the OPE ones. Therefore, to judge the predictive power of OPT($\pi$), this difference, and not the phase parameter itself has to be judged.

## IV. THEORETICAL TESTS

In this section we judge OPT and OPT($\pi$) from a theoretical side, analogous to the test in Ref. [12]. The higher p.w. phase parameters predicted by OPT and OPT($\pi$) with the signature mapping from the lower p.w. phase parameters of a realistic $NN$ potential model are compared with the higher p.w. phase parameters of that model. Furthermore we examine the significance of the differences in higher p.w. phase parameters between OPE, OPT and this potential model in the description of the scattering data.

The potential model considered is

$$V = V_C + V^N_{OPE} + V^N_{HBE} \; . \tag{11}$$

Here $V_C$ is the point-Coulomb potential, $V^N_{HBE}$ is the non-OPE part of the Nijmegen soft core potential [13], and $V^N_{OPE}$ is the OPE potential with a form factor as in the Nijmegen potential

$$V^N_{OPE} = f_0^2 \, \frac{M_p}{E} \left(\frac{m}{m_{\pi^+}}\right)^2 m \left[\frac{1}{3}\left(\vec{\sigma}_1 \cdot \vec{\sigma}_2\right) \phi^1_C + S_{12}\phi^0_T\right] \; , \tag{12}$$

with $m$ the $\pi^0$-mass, and $E$ the c.m. total energy of a proton. The functions $\phi^1_C$ and $\phi^0_T$ are as defined in Ref. [13]. The pseudo-vector $pp\pi^0$-coupling constant $f_0^2$ has been determined from the 0–350 MeV $pp$ scattering data in Ref. [21] as $f_0^2 = (72.5 \pm 0.6)\cdot 10^{-3}$.

For each $\omega_i$ the lowest p.w. phase parameters of the potential are input. In the case of OPT($\pi$), the OPE amplitude $\tilde{\omega}$ subtracted is the unitarized OPE amplitude in Coulomb-distorted-wave Born approximation. OPT or OPT($\pi$) provide higher p.w. phase parameters that can be compared with the phase parameters of the model. As stated before, in the case of OPT($\pi$) it is the difference of the predicted phase parameter with the OPE phase parameter, that is to be judged. In the case of OPT, $x_c$ is given by the Coulomb + OPE branchpoint. Since this branchpoint is removed when we use OPT($\pi$), the nearest branchpoint then corresponds to two-pion-exchange.

The predictions of OPT and OPT($\pi$) have been computed for all partial waves with $J < 10$. The results for some partial waves are given in Fig. 4. The predictions are labeled OPT$n$ and OPT($\pi$)$n$, where $n$ is the number of input phase parameters for the predictions. For the $^1G_4$ and the $^3F_4$ only $n = 1$ is possible, since the $^1S_0$ and the $^3P_0$ have no predictive power. Also for the partial waves not displayed the OPT predictions are always in the correct direction, which demonstrates the predictive power of OPT. Of course, this is of no practical use in a phase shift analysis, since the C+OPE phase parameters are in general a far better approximation to the phase parameters of the model. In general OPT($\pi$) works well for the $^1S_0$-, $^3P_1$-, and $^3P_2$-sequences. However, it does *not* work for the mixing parameters and the $^3P_0$-sequence.

The simplest interpretation for these mispredictions could be that we work with heavily truncated expansions, so we cannot expect very good results, but the origin of these mispredictions can be seen in the model phase parameters: both the $\varepsilon_2$ and $^3F_2$ of the model deviate very much from OPE, but the higher p.w. phase parameters of these sequences deviate only slightly from OPE. Therefore no singularity other than OPE can be of importance here. OPT($\pi$) will extrapolate the large deviation from OPE in the $\varepsilon_2$ and $^3F_2$ smoothly to higher $J$ and therefore predict wrong higher partial waves.

At 300 MeV in the cases where OPT($\pi$) works well, the difference of the model phase shifts with the C+OPE phase shifts decreases with a factor of 5 to 10 if $\ell$ is increased by 2, but decreases with a factor of 80 or larger from the $\varepsilon_2$ to the $\varepsilon_4$, and from the $^3F_2$ to the $^3H_4$. Had we taken the parametrized Paris potential [22] instead of the Nijmegen potential, then the predictions for higher mixing parameters would have been more in accordance with the model values, but the predictions in the $^3P_0$-sequence would still be incorrect. That the higher mixing parameters are then predicted better is probably caused by the fact that the model values differ more from OPE.

The conclusion to be drawn from the tests with potential models, as displayed in Fig. 4, is that OPT and OPT($\pi$) work well if there is a definite trend in the phase parameters or in the deviation from OPE of the phase parameters.

The importance of the differences in the higher p.w. phase parameters can probably best be measured via the $\chi^2_{\rm min}$ reached when compared with the experimental data. In a phase shift analysis as discussed in the next section, roughly the phase shifts up to and including $\ell = 4$ and the mixing parameters $\varepsilon_2$ and $\varepsilon_4$ are parametrized. Therefore we have computed for several models for the higher p.w. phase parameters, the $\chi^2_{\rm min}$ reached on the 874 scattering data that we have between 30 and 350 MeV. The nuclear interaction in the lower partial waves was taken to be the Nijmegen soft core potential [13]. The $\chi^2_{\rm min}$ obtained in each case is given in Table III.

The results in Table III are not what one would expect. It is clear that no interaction in the higher partial waves is the least realistic model, but here it appears that *all* OPT predictions (not only when the difference with the OPE phase parameters is predicted) are better than the OPE and even the NSC phase parameters. From Fig. 4 one would have expected the OPT method to be worse than OPE. In the next section we will see that the performance in a phase shift analysis of the OPT methods is somewhat different. There the lower partial waves will be adjusted when taking different prescriptions for the interaction in the higher partial waves. Furthermore, the $\chi^2$ attained in the analysis will be much closer to the expected $\chi^2_{\rm min}$/d.o.f.= 1.

## V. TESTS IN THE PHASE SHIFT ANALYSIS

A realistic application of OPT is its use in a phase shift analysis. We present here some results of an analysis of all *pp* scattering data in the energy range $0 < T_{\rm lab} < 350$ MeV [23]. Analogous to our previous analyses of *pp* scattering data [24,21] we used an energy-dependent $P$-matrix parametrization to parametrize the short range interaction. The long range interaction is described by a potential tail. The electromagnetic part consists of the modified relativistic Coulomb potential [25] and the vacuum polarization potential [26]. For the nuclear part of the potential tail we chose either the OPE potential or the OPE potential plus the heavier boson exchange parts of the Nijmegen potential [13]. As the OPE potential is only needed here for $r > b$, we can neglect the form factor that enters Eq. 12 via $\phi_C^1$ and $\phi_T^0$. The parameters in the phase shift analysis are the pion-coupling constant and a number of $P$-matrix parameters.

One can parametrize only a finite number of phase parameters in an analysis. Those that are not parametrized are what actually interests us here. We compare three different choices for these higher p.w. nuclear phase parameters (i) OPE phase parameters, computed in Coulomb-distorted-wave Born approximation, (ii) the OPT($\pi$) predictions, as described earlier, and (iii) the phase parameters of the Nijmegen soft core potential.

For these three choices, we made fits to the data using different numbers of parameters. In Table IV we show how the $P$-matrix parameters were distributed over the different partial waves. For each total number of parameters the same parametrization was used in the lower partial waves, independent of how the higher p.w. phase parameters were taken. This guarantees that every effect seen is due to the choice of higher p.w. phase parameters, although it does not ensure the lowest possible $\chi^2_{\rm min}$ with a given total number of parameters for all three choices.

The obtained $\chi^2_{\rm min}$/d.o.f. where for the lower partial waves the OPE potential tail was taken is shown Fig. 5a, that for the OPE+NSC potential tail in Fig. 5b. Since we consider two descriptions of the potential tail, we test in fact the OPT($\pi$) prediction scheme for two different physical models. In order to make visible both the large differences in $\chi^2_{\rm min}$ for few parameters and the much smaller differences for about 30 parameters, the vertical scale in Fig. 5 is chosen as log ($\chi^2_{\rm min}$/d.o.f. $-1$).

Apart from the simple observation that with more parameters (and therefore more parametrized phase parameters) a better treatment of higher partial waves grows less important, we can draw several conclusions from the results in Fig. 5. OPT($\pi$) is better than OPE for the higher partial waves. The Cutkosky and Deo and signature mappings are comparable in predictive power. The Nijmegen potential model provides better higher p.w. phase parameters than OPT($\pi$). For the 29 parameter fits, the higher p.w. phase parameters of the Nijmegen potential give $\chi^2_{\rm min}$ about 20 less than the OPE higher p.w. parameters, and the OPT higher p.w. parameters give a $\chi^2_{\rm min}$ inbetween. Also in the lower partial waves the tail of the Nijmegen potential is seen to be better than the OPE-tail, $\Delta\chi^2_{\rm min} \approx 30$ for the 29 parameter fits.

Guided by the results of the preceding section, we also investigated whether OPT($\pi$) can be seen to mispredict the phase parameters in certain sequences of partial waves. In the theoretical tests of Section IV, OPT($\pi$) failed to predict the ${}^3P_0$-sequence, and the quality of prediction of the mixing parameters was at best meager. In the other p.w. sequences OPT($\pi$) correctly predicted the higher p.w. phase shifts. To have an analogous test in a fit to the data, we used OPT($\pi$) only for 4 of the 5 p.w. sequences, thereby excluding each p.w. sequence one at a time. This has been done for the 29 parameter fit with the OPE+NSC potential tail. The results are given in Table V, from which one can see that predicting the ${}^1S_0$ and ${}^3P_1$-sequences and *not* predicting the ${}^3P_0$-sequence improves the fit, which is in agreement with the result of the theoretical test.

## VI. DISCUSSION

We have analyzed the importance of the use of optimal mapping techniques to describe the higher partial wave phase parameters in the analysis of 0–350 MeV *pp* scattering data.

Tests within a potential model as well as tests in the phase shift analysis show that OPT can in principle correct for a substantial part of the error made due to the truncation of the p.w. series. For the ${}^3P_0$-sequence and for the mixing parameters the method fails to produce good results. Within the potential model this can be related to the non-smooth behavior of the expansion coefficients (phase parameters).

Inclusion of OPT in the phase shift analysis (at least for a subset of the p.w. series) gives a good and model-independent description of the higher p.w. phase parameters. The higher p.w. phase parameters of the Nijmegen soft core potential [13] appear to be even better than those predicted by OPT. This indicates that the non-OPE part of the Nijmegen potential contains valuable physics, which cannot be described adequately enough by OPT. The quality of the higher p.w. phase parameters of other $NN$ potentials could be tested in the same way.

For a high-quality phase shift analysis the inclusion of OPT techniques in its simplest form, as employed here, appears to be not very important for low energy *pp* scattering ($\Delta\chi^2_{\rm min} \approx 10$). Clearly, in an analysis where less partial waves can be parametrized, the use of OPT can improve the description of the scattering data dramatically. Therefore the use of OPT will certainly be important in the analysis of higher energy data or incomplete data sets. Further study deserves the question whether a more refined and sophisticated application of OPT, based on Cutkosky's theory [15,27] on the representation of scattering data by analytic functions, will improve the phase shift analysis. This we leave for a future combined phase shift analysis of the *pp* and *np* data.

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## APPENDIX A

In our applications we used the optimal mapping for two different singularity structures. The mapping for an $x$-plane with two cuts, along $(-\infty, -x_c]$ and $[x_c, \infty)$, will be denoted as $z_{\rm sym}(x_c; x)$ and the mapping for the $x$-plane with only the right-hand cut as $z_{\rm as}(x_c; x)$.

From the theory of conformal mappings it is known that the function that maps a specific region on to a specific image contains three arbitrary real constants. The image of our mappings is only demanded to be a unifocal ellipse, of which the size can still vary. To make the mapping unique, we have to supply four real, or two complex constants. This is done by requiring that $z(x = \pm 1) = \pm 1$. The whole problem has complex conjugation symmetry. Therefore the mapping will be a real-analytic function and the interval $[-1, 1]$ will be mapped on to itself, since its endpoints are invariant.

For the construction of the mapping $z_{\rm sym}(x)$ we follow Cutkosky and Deo [3], where it is given as

$$z_{\rm sym}(x_c; x) = \sin\left[\frac{\pi}{2}\frac{F(\sin^{-1} x, 1/x_c)}{F(\pi/2, 1/x_c)}\right] ,$$

where $F(\phi, k)$ is the incomplete elliptic integral of the first kind [28]. They also pointed out that using repeated Gauss transformations leads to a very simple computation scheme. We only summarize the results. First define $v_0(x) = x$ and $k_0 = 1/x_c$. Then use repeatedly

$$v_{n+1}(x) = v_n(x)\frac{1 + \sqrt{1 - k_n^2}}{1 + \sqrt{1 - k_n^2 v_n^2(x)}} \quad \text{and} \quad k_{n+1} = \frac{k_n^2}{1 + \sqrt{1 - k_n^2}} .$$

Now one obtains $\lim_{n\to\infty} v_n = z_{\rm sym}(x_c; x)$. It is easily seen that the convergence is very rapid, since $k_n$ converges to zero quadratrically.

The mapping $z_{\rm as}$ can be expressed in $z_{\rm sym}$ as

$$z_{\rm as}(x_c; x) = -1 + 2\left[z_{\rm sym}\left(\sqrt{\frac{x_c + 1}{2}}; \sqrt{\frac{x + 1}{2}}\right)\right]^2 .$$

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# TABLES

| $n$ | | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|
| exact | $a_n$ | 1.036 | 0.556 | 0.290 | 0.160 | 0.086 | 0.046 | 0.025 |
| linear | $a_n$ | 1.036 | 0.556 | 0 | 0 | 0 | 0 | 0 |
| OPT | $\tilde{a}_n$ | 1.022 | 0.639 | 0.229 | 0.092 | 0.039 | 0.018 | 0.008 |
| OPT′ | $\tilde{a}_n$ | 1.036 | 0.556 | 0.199 | 0.080 | 0.034 | 0.015 | 0.007 |

TABLE I. Exact expansion coefficients of our example function and approximations to them. Linear: Two expansion parameters determined with the $L_2$ inner product. OPT: Two OPT expansion parameters determined with the $L_2$ inner product. OPT′: Two OPT expansion parameters determined to reproduce the first two $a_n$.

| amplitude | $\omega_1$ | $\omega_{11}$ | $\omega_{22}$ | $\omega_{12}$ |
|---|---|---|---|---|
| signature | odd | even | even | even |
| content | ${}^3P_1$, ${}^3F_3, \ldots$ | ${}^3P_2$, ${}^3F_4, \ldots$ | ${}^3P_0$, ${}^3F_2, \ldots$ | $\varepsilon_2, \varepsilon_4, \ldots$ |

TABLE II. The spin-triplet amplitudes which are expanded (see text).

| Model for higher phase parameters | none | OPE | NSC | OPT | OPT($\pi$) |
|---|---|---|---|---|---|
| $\chi^2_{\text{min}}$ | 3636.2 | 1937.4 | 1855.1 | 1806.0 (s)<br>1820.7 (C) | 1771.5 (s)<br>1768.4 (C) |

TABLE III. $\chi^2_{\text{min}}$ on the 874 scattering data between 30 and 350 MeV of the NSC potential model plus the point Coulomb interaction, with different models for the higher p.w. wave phase parameters. OPE: with $f_0^2 = 73 \cdot 10^{-3}$; s: signature mapping; C: Cutkosky and Deo mapping.

| $^1S_0$ | $^3P_0$ | $^3P_1$ | $^3P_2$ | $^1D_2$ | $\varepsilon_2$ | $^3F_2$ | $^3F_3$ | $^3F_4$ | $^1G_4$ | $\varepsilon_4$ | $^3H_4$ | total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 2 | 2 | 3 | 2 | 1 | - | - | - | - | - | - | 13 |
| 3 | 2 | 2 | 3 | 3 | 1 | - | - | - | - | - | - | 15 |
| 4 | 2 | 2 | 3 | 3 | 1 | - | - | 2 | - | - | - | 18 |
| 7 | 3 | 2 | 3 | 3 | 1 | - | - | 2 | - | - | - | 22 |
| 7 | 3 | 2 | 3 | 3 | 1 | 1 | 1 | 2 | - | 1 | - | 25 |
| 7 | 3 | 2 | 3 | 3 | 1 | 1 | 2 | 2 | 2 | 1 | - | 28 |
| 7 | 3 | 3 | 3 | 3 | 1 | 1 | 2 | 2 | 2 | 1 | - | 29 |
| 7 | 3 | 4 | 3 | 3 | 1 | 1 | 3 | 3 | 2 | 1 | - | 32 |
| 7 | 3 | 4 | 3 | 3 | 1 | 1 | 3 | 3 | 2 | 1 | 1 | 33 |

TABLE IV. Distribution of the parameters over the different partial waves. The total number of parameters also includes the pion-coupling constant.

| partial wave sequence excluded | $^1S_0$ | $^3P_1$ | $^3P_2$ | $\varepsilon_2$ | $^3P_0$ |
|---|---|---|---|---|---|
| $\Delta\chi^2_{\text{min}}$ | 7.6 | 4.4 | 0.8 | $-0.2$ | $-9.0$ |

TABLE V. Change in $\chi^2_{\text{min}}$ when OPT($\pi$) is used for a subset of the partial waves.

## FIGURES

FIG. 1. The optimal mapping for $x_c = 1.2$.

FIG. 2. The example function $f(x) = (x_c - x)^{-1/2}$ (full line) and the two term expansion approximations to it. Dashed: without OPT. Dotted: with OPT.

FIG. 3. Singularity structure of the spin-singlet nuclear amplitude $M_S$ in the complex $x = \cos\theta$-plane.

FIG. 4. Model and C+OPE ($\pi$) phase parameters in degrees compared with predicted values. OPT$n$: prediction of OPT with $n$ lower phase parameters as input. OPT($\pi$)$n$: predictions of OPT($\pi$) with $n$ lower phase parameter differences as input. Pion-coupling constant used $f_0^2 = 73 \cdot 10^{-3}$.

FIG. 5. $\chi^2_{\rm min}$/d.o.f. versus number of parameters. (a): lower partial waves described with the OPE potential tail. (b): lower partial waves described with the OPE+NSC potential tail. Dashed: higher p.w. phase parameters OPE. Dotted: higher p.w. phase parameters OPT($\pi$) (signature mapping). Dashed-dotted: higher p.w. phase parameters OPT($\pi$) (Cutkosky and Deo mapping). Full line: higher p.w. phase parameters of the Nijmegen potential.