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Soft Two-Pion-Exchange Nucleon-Nucleon Potentials* †

Th.A. Rijken

Institute for Theoretical Physics, University of Nijmegen, Nijmegen, the Netherlands

Abstract

Two-Pion-Exchange nucleon-nucleon potentials are derived for the Pseudo-Vector pion-nucleon interaction, assuming strong dynamical pair-suppression. At the pion-nucleon vertices we include Gaussian form factors, which are incorporated into the relativistic two-body framework by using a dispersion representation for the One-Pion-Exchange amplitude. The Fourier transformations are performed using a factorization technique for the energy denominators. This leads to analytic expressions for the TPE-potentials containing at most one-dimensional integrals. The TPE-potentials are calculated up to orders f^4 and $(m/M)f^4$. The terms of order f^4 come from the adiabatic contributions of the parallel and crossed three-dimensional momentum-space TPE-diagrams, and from the non-adiabatic contributions of the OPE-iteration. The (m/M)-corrections are due to the $1/M$-terms in the non-adiabatic expansion of the nucleon energies in the intermediate states, and the $1/M$-terms in the pion-nucleon vertices. The latter are typical for the PV-coupling and would be absent for the PS-coupling. The Gaussian form factors lead to soft TPE-potentials. These potentials can readily be exploited in NN-calculations in combination with *e.g.* the Nijmegen soft-core OBE-model, and in nuclear (matter) calculations.

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I. INTRODUCTION

Since the forties there has been much activity in calculating the Two-Pion-Exchange (TPE) corrections to the One-Pion-Exchange (OPE). An excellent review of the literature before 1960 as well as an expert survey of the physics involved can be found in [1]. Two-Pion-Exchange-Potential's (TPEP's) with Gaussian form factors have not been dealt with systematically in the literature. They have, however, been used in the fifties in the Gartenhaus-potential [2]. More recently, they have been exploited extensively in One-Boson-Exchange (OBE) potentials for nucleon-nucleon [3] and hyperon-nucleon [4] scattering. These so-called soft-core OBE-models are based on the Reggeon picture of hadrons and hadronic interactions [5,6], where Gaussian form factors emerge most naturally.

In [7], we have argued that the soft-core OBE-model is qualitatively in accordance with the important modern viewpoints in strong interactions: Regge-phenomenology, the QCD-picture, the non-relativistic quark-model, and chiral-symmetry. For example, it is consistent with the results of the soft-pion theorems for low-energy pion-nucleon scattering [8]. The large contribution to the πN-scattering-length a_0^+ from ε-exchange is cancelled by the contribution from Pomeron-exchange, and no nucleon-antinucleon pair-terms are needed.

In this paper we use these modern viewpoints as general guidelines and derive TPEP's with Gaussian form factors. We show in detail how Gaussian form factors can be incorporated in TPEP's, starting from the relativistic interaction theories. Also, the objective of this work is to prepare an extension of the OBE-model of ref. [3], where the Two-Pion-Exchange-Potential (TPEP) is included. Hence, we derive configuration space TPEP's for the Schrödinger equation. The results are TPEP's that are soft at small distances, and which can be used not only in nucleon-nucleon, but *e.g.* also easily in nuclear and nuclear-matter calculations.

An important issue has always been whether or not nucleon-antinucleon pairs are dynamically suppressed [9] in the pion-nucleon or, more general, in the meson-nucleon interaction. This so-called 'pair-suppression' would imply that the nucleon off-mass-shell effects are small. From the point of view of the non-relativistic quark-model, pair-suppression seems rather natural [10]. Moreover, it is supported by large-N considerations in QCD [11]. Phenomenologically, for the pion-nucleon interaction we have at our disposal the PS-coupling and the PV-coupling. The PS-coupling means strong pair-terms, whereas the PV-coupling means weak pair-terms. Whether with a simple PV-coupling the pair-suppression is realized sufficiently is an open question. As a working hypothesis, we assume that there is a strong dynamical pair-suppression and leave out all pair-contributions to the TPEP. In effect, it practically amounts to the choice of the PV-coupling. For the latter, the one-pair and the two-pair terms are rather weak, they are respectively of order $(m/M)f^4$ and $(m/M)^2f^4$. (Given the general arguments for pair-suppression above, we guess this to be true for all meson-nucleon interactions [12].)

The foregoing remarks fit in very well with the picture which emerges from the effective pion-nucleon interaction in the non-linear realization of chiral-symmetry [13,14]. When the ρ-meson is included in the chiral-symmetric Lagrangian, there no longer appears a scalar-coupling quadratic in the pion field. As is well known strong quadratic coupling is characteristic for the PS-coupling after an equivalence transformation [15]. In fact, the expansion of the Schwinger-Weinberg Lagrangian [14], keeping in principle terms to second order in the pion field, gives only a PV-coupling linear in the pion field. In tree-approximation, this Lagrangian satisfactorily accounts for the low-energy parameters of the reactions involving pions.

The derivation of the TPEP starts from the relativistic two-body equation [17,19], where the interaction kernel is given by the two-nucleon-irreducible Feynman diagrams. These are the diagrams with at most two-pions in the intermediate states. We apply the procedure of Salpeter [20] to the relativistic two-body (two-nucleon) equation by performing the energy integrations. This leads to the three-dimensional integral equation of Thompson [21], a definition of the interaction kernel, and a definition of the wave-function.

The particular procedure we use, was given by Klein [22,23]. Here, in principle, an *ansatz* is made for the relative-energy dependence of the two-body wave-function. Since however, in the derivation of the TPEP we may restrict ourselves to reproduce the Feynman graphs up to order f^4, we apply in reality the Klein *ansatz* only to the free two-body wave-function, where this is easily seen to be correct. In doing the energy integrations, we avoid the occurrence of more than one energy integration-variable in any propagator of the Feynman integrals. This is achieved by introducing, where necessary, new variables using δ-functions and using for the latter a familiar integral representation. This way we arrive at the 'old-fashioned perturbation' diagrams in a straigthforward and unambiguous way. (This is in contrast to the evaluation of the energy-integrals in the original paper [22]).

The procedures, indicated above, are first carried out for pointlike vertices. Then, we generalize the results for the presence of the Gaussian formfactors. In doing this, we employ the dispersion representation of the OPE-amplitude. The generalization on the level of the 'old-fashioned perturbation' diagrams is then rather straightforward. This seems the only practical way to incorporate Gaussian form factors. To do the calculations directly for Feynman diagrams is not so simple in that case. For instance, the Feynman integrals for the box-diagrams with form factors, which have a Gaussian behavior for space-like momentum transfers, can not be evaluated in the usual way.

In carrying through the analytic derivation of our formulas, we improved a technique already used by Levy [24], which enables us to express the potentials in one-dimensional integrals over products of the OPE-functions already given in [3].

The diagrams which we calculate are: (i) the parallel and crossed TPE-diagrams that have been calculated *e.g.* by Brueckner and Watson [25]; and (ii) the iterated OPE-diagrams calculated by Taketani, Machida, and Ohnuma [26]. Although these calculations are generalized, using Gaussian form factors at the vertices instead of point-couplings, we still refer in the following to the corresponding potentials as the BW- and TMO-potentials. Although pair-diagrams are totally absent in this work, our methods are, of course, fully adequate to treat the pair-diagrams which have occurred in the literature [25,27,28].

The potentials are calculated up to orders f^4 and $(m/M)f^4$. The leading terms are the so-called 'adiabatic' ones [29]. The (m/M)-corrections come from (i) $1/M$-terms of the pion-nucleon interaction ('vertex-contributions'), and (ii) from the $1/M$ recoil terms in the energy denominators of the intermediate states ('non-adiabatic contributions'). The terms of still higher order in m/M we neglect. They become comparable with f^6-terms, which correspond to the three-pion exchange potentials. The latter are not considered in this paper.

In carrying through the calculations, we have ignored purely off-energy-shell contributions to the potentials. In principle these could be included as well, but this would make the algebra more cumbersome.

In defining the PV-coupling constant we follow [30] and take as scaling mass the charged pion mass, denoted by m_π. We do not distinguish between the different nucleon masses (denoted by M) or between the different pion masses and coupling constants. So our results are SU(2)-symmetric. The (average) pion-mass is denoted by m. In sections III - VII we leave out the isospin indices. In sections VIII and IX the proper isospin factors are added, *i.e.* $2-3\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2$ and $2+3\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2$ for respectively the planar and the crossed diagrams.

The contents of this paper is arranged into ten sections and five appendices. Section II reviews the pion-nucleon interaction in the context of the Schwinger-Weinberg Lagrangian. Here we also make some remarks on the form factor and discuss a covariant phenomenological description of dynamical pair-suppression. In Sections III and IV the general approach within the framework of relativistic quantum mechanics is presented. Especially the connection between the relativistic two-body equation description and that of the three-dimensional formalism is reviewed. In Section V and Appendix A the One-Pion-Exchange and Two-Pion-Exchange kernels are derived for point-interactions. In Section VI and Appendix B, the procedure to include the Gaussian form factor is described. In particular, in Appendix B the factorization technique for the energy-denominators is given. In Section VII and Appendix C the potential for the Lippmann-Schwinger equation is defined and the expansion of the energy denominators for the intermediate states are discussed. In Section VIII and IX, using Appendices B and D, the TPEP for the BW-graphs and TMO-graphs are derived. In Section X we give the results and compare the TPEP's of this work with the literature. We use for that purpose the corresponding potentials in the point-coupling limits, which are given in Appendix E. Also we discuss briefly the (minor) changes in the TPEP that have to be made, if one assumes PS-coupling in combination with strong dynamical pair-suppression.

II. THE PION-NUCLEON INTERACTION

A major development in the theory of the pion-nucleon interaction has been the construction of SU(2)×SU(2) chiral-symmetric Lagrangians, see *e.g.* [13,14]. Considering the two ways of chiral-symmetry realization, the linear and the non-linear one, the second seems the most natural. This because in QCD, broken-chiral-symmetry is realized in the Nambu-Goldstone mode [31]. The pion is supposed to play the role of the Golstone boson connected with the SU(2)×SU(2)-symmetry breaking of the vacuum. In the linear realization of chiral-symmetry one has PS-coupling and one needs a σ-field, *e.g.* in order to generate the nucleon mass. However such a particular link between chiral-symmetry, the nucleon mass and eventually the scalar mesons seems rather naive and too restrictive. Therefore we choose for the discussion of the pion-nucleon-vertex the effective Lagrangian in the nonlinear realization, which has been constructed by Schwinger and Weinberg [14]. As mentioned in the Introduction, this Lagrangian satisfactorily accounts for the low-energy parameters of reactions with pions.

In linear approximation, local chiral-symmetry leads to the effective Lagrangian [14]

$$\mathcal{L} = i\bar{\psi}'\gamma_\mu\mathcal{D}^\mu\psi' - M\bar{\psi}'\psi' - \frac{f}{m_\pi}\bar{\psi}'\gamma_5\gamma_\mu\boldsymbol{\tau}\psi'\cdot(D^\mu\boldsymbol{\varphi} - m_A A^\mu) \tag{2.1}$$

where ψ' is the nucleon field, m_π is the charged pion mass, f is the pion-nucleon PV-coupling constant [32], and A_μ is the axial-vector meson field. In the Lagrangian (2.1) the chiral-covariant-derivatives are given by [33]

$$\mathcal{D}_\mu = \partial_\mu + \frac{i}{2}\, g\boldsymbol{\tau}\cdot\boldsymbol{\rho}_\mu \ , \qquad D_\mu\boldsymbol{\varphi} = \Big(\partial_\mu + ig\boldsymbol{\rho}_\mu\times\boldsymbol{\varphi}\Big) \ . \tag{2.2}$$

Here $g = m_\rho/(\sqrt{2}f_\pi)$, where f_π is the pion-decay constant, which is related to the pion-nucleon coupling f by the Goldberger-Treiman relation [34]

$$\frac{1}{2f_\pi} = \left|\frac{G_V(0)}{G_A(0)}\right|\left(\frac{f}{m_\pi}\right) \ . \tag{2.3}$$

where $|G_A/G_V| \approx 1.26$ and $f_\pi \approx 93$ MeV [30].

The interaction terms in the Lagrangian (2.1) describe the low energy s-wave pion-nucleon scattering very well. For a closer discussion of these terms see [14].

Moreover, the Lagrangian (2.1) is part of a Lagrangian (see *e.g.* [35] for full details), which represents a non-linear realization of SU(2)×SU(2) chiral-symmetry and which describes also the very low energy pion-pion.

Since our interest is restricted to the TPEP, we keep only terms which contain the pion field. The remaining pion-nucleon interaction Lagrangian is then solely the PV-coupling term

$$\mathcal{L}_I = -\frac{f}{m_\pi}\bar{\psi}\gamma_5\gamma_\mu\boldsymbol{\tau}\psi\cdot\partial^\mu\boldsymbol{\varphi} \ . \tag{2.4}$$

When the πNN-form factor $F(\mathbf{x}'-\mathbf{x})$ is included, the interaction density becomes modified

$$L_I(\mathbf{x}) = \int d^3x'\, F(\mathbf{x}'-\mathbf{x})\mathcal{L}_I(\mathbf{x}') \ . \tag{2.5}$$

The potentials in momentum space are the same as for point interactions, except that the coupling constants are multiplied by the Fourier transform $F(\mathbf{k}^2)$ of the form factor, where $\mathbf{k}$ is the momentum transfer at the πNN-vertex. We use for space-like momentum transfers a Gaussian parameterization of the form factors in the OPE-amplitudes, *i.e.* $F(\mathbf{k}^2) \approx \exp(-\mathbf{k}^2/\Lambda^2)$. Until section VI we treat the point-coupling limit of the interactions, which makes the discussion less complicated. In section VI we implement the Gaussian form factors by employing a dispersion representation for the OPE-amplitude, valid for all momentum transfers. The interaction Hamiltonian density is

$$\mathcal{H}_I = (f/m_\pi)\ \bar{\psi}\gamma_5\gamma_\mu\boldsymbol{\tau}\psi\cdot\partial^\mu\boldsymbol{\varphi} - \frac{1}{2}(f/m_\pi)^2(\psi^*\gamma_5\boldsymbol{\tau}\psi)^2 \ . \tag{2.6}$$

The last term in the Hamiltonian is the so-called 'contact'-term, whose appearance is a general feature of theories where the interaction Lagrangian contains derivatives. It gives no contributions to the Green functions or the potential, since it is compensated by the noncovariant piece in the contraction

$$\langle 0|T(\partial_\mu\varphi(x)\partial_\nu\varphi(y))|0\rangle = \partial^x_\mu\partial^y_\nu\ \langle 0|T(\varphi(x)\varphi(y))|0\rangle - ig_{\mu 0}g_{\nu 0}\delta^4(x-y) \ . \tag{2.7}$$

As stated in the Introduction, we use the hypothesis that pair-suppression is operating in nature. Therefore, we neglect the transitions between positive- and negative-energy solutions. Phenomenologically, a covariant form of pair-suppression can be introduced easily in principle. For example, consider the πNN-vertex in momentum space

$$\Gamma_5'(p',p) = \exp[a(\gamma p' - M)]\Gamma_5(p',p)\ \exp[a(\gamma p - M)] \ , \tag{2.8}$$

where a is a free parameter and $\Gamma_5(p',p)$ is the PV- or the PS-vertex. By taking a large and positive, the off-mass-shell effects can be suppressed strongly. In that case, the pair-terms i.e. the so-called Z-diagrams can be neglected. Of course, the QCD and non-relativistic quark-model arguments in favor of this, are quite general and hold also for the vector-, scalar-, etc. mesons. In that situation, a calculation with the Thompson-equation, neglecting the coupling to negative-energy states, becomes more physical than a calculation with the Bethe-Salpeter equation without off-mass-shell suppression.

Based on the discussion of the πNN-vertex in the Introduction and in this section, we come to the following aproximate treatment of pair-suppression. We do not use a form of a type similar to (2.8) explicitly. Instead, we use the simple PV-coupling but neglect the transitions between the positive and negative energy states.

III. RELATIVISTIC TWO-BODY EQUATION

We consider the nucleon-nucleon reaction

$$N_a(p_a,s_a) + N_b(p_b,s_b) \rightarrow N_a'(p_a',s_a') + N_b'(p_b',s_b') \ . \tag{3.1}$$

Introducing the total and relative four-momentum for the initial and final state

$$\begin{array}{ll} P = p_a + p_b & ,\ P' = p_a' + p_b' \ , \\ p = \frac{1}{2}(p_a - p_b) & ,\ p' = \frac{1}{2}(p_a' - p_b') \ , \end{array} \tag{3.2}$$

we have in the center-of-mass system (cm-system) for a and b on-mass-shell

$$P = (W,\mathbf{0}) \ ,\ \ p = (0,\mathbf{p}) \ ,\ \ p' = (0,\mathbf{p}') \ . \tag{3.3}$$

In general, the particles are off-mass-shell in the Green-functions. In the following, the on-mass-shell momenta for the initial and final states are denoted respectively by p_i and p_f. So, $p_{i0} = E(\mathbf{p}_i) = \sqrt{\mathbf{p}_i^2 + M^2}$ and $p_{f0} = E(\mathbf{p}_f) = \sqrt{\mathbf{p}_f^2 + M^2}$. Because of translation-invariance $P = P'$ and $W = W' = 2E(\mathbf{p}_i) = 2E(\mathbf{p}_f)$.

The two-body relativistic scattering-equation reads

$$\psi(p,P) = \psi^0(p,P) + G(p;P)\int d^4p'\ I(p,p')\ \psi(p',P) \ , \tag{3.4}$$

where $\psi(p,P)$ can be expressed as a 4×4 matrix in Dirac-space, the interaction kernel is denoted by I, and G is the two-particle Green-function. The contributions to the kernel I come from the two-nucleon irreducible Feynman diagrams. For the inhomogeneous term we have $\psi^0(p') \sim \delta^4(p'-p_i)$. In writing (3.4) we have separated off an overall δ-function describing the conservation of the total four-momentum.

The two-particle Green-function is

$$G(p;P) = \frac{i}{(2\pi)^4}\left[\frac{1}{\gamma(\frac{1}{2}P+p)-M}\right]^{(a)}\left[\frac{1}{\gamma(\frac{1}{2}P-p)-M}\right]^{(b)} . \tag{3.5}$$

The projection operators $\Lambda_+(\mathbf{p})$ and $\Lambda_-(\mathbf{p})$ on the positive- and negative-energy states are

$$\Lambda_+(\mathbf{p}) = \sum_s u(\mathbf{p},s)\otimes\bar{u}(\mathbf{p},s) \ , \ \ \Lambda_-(\mathbf{p}) = -\sum_s v(\mathbf{p},s)\otimes\bar{v}(\mathbf{p},s) \ . \tag{3.6}$$

For particles on the mass shell, *i.e.* for which $p_0 = E(\mathbf{p})$, one has $2M\Lambda_\pm(\mathbf{p}) = M \pm \not{p}$. The propagator of a spin-$\frac{1}{2}$ particle off the mass shell can be expressed as follows

$$\frac{\not{p}+M}{p^2-M^2+i\delta} = \frac{M}{E(\mathbf{p})}\left[\frac{\Lambda_+(\mathbf{p})}{p_0-E(\mathbf{p})+i\delta} - \frac{\Lambda_-(-\mathbf{p})}{p_0+E(\mathbf{p})-i\delta}\right] . \tag{3.7}$$

Therefore, in the cm-system, where $\mathbf{P}=0$ and $P_0 = W$, the Green-function can be written as

$$\begin{aligned} G(p;W) = \frac{i}{(2\pi)^4}\left[\frac{M}{E(\mathbf{p})}\right]^2 \cdot &\left[\frac{\Lambda^a_+(\mathbf{p})}{W/2+p_0-E(\mathbf{p})+i\delta} - \frac{\Lambda^a_-(-\mathbf{p})}{W/2+p_0+E(\mathbf{p})-i\delta}\right] \\ \times &\left[\frac{\Lambda^b_+(-\mathbf{p})}{W/2-p_0-E(\mathbf{p})+i\delta} - \frac{\Lambda^b_-(\mathbf{p})}{W/2-p_0+E(\mathbf{p})-i\delta}\right] . \end{aligned} \tag{3.8}$$

Multiplying this out we write the ensuing terms in shorthand notation

$$G(p;W) = G_{++}(p;W) + G_{+-}(p;W) + G_{-+}(p;W) + G_{--}(p;W) \ , \tag{3.9}$$

where G_{++} etc. corresponds to the term with $\Lambda^a_+\Lambda^b_+$ etc. Introducing the wave-functions (see [20])

$$\psi_{rs}(p') = \Lambda^a_r\Lambda^b_s\psi(p') \qquad\qquad (r,s=+,-) \ , \tag{3.10}$$

the two-body equation (3.4) can be written for *e.g.* ψ_{++} as

$$\begin{aligned} \psi_{++}(p) = \psi^0_{++}(p) + G_{++}(p;W) \int d^4p' \cdot \\ \times [I(p,p')_{++,++}\psi_{++}(p') + I(p,p')_{++,+-}\psi_{+-}(p') \\ + I(p,p')_{++,-+}\psi_{-+}(p') + I(p,p')_{++,--}\psi_{--}(p')] \ . \end{aligned} \tag{3.11}$$

Invoking now 'dynamical pair-suppression', as discussed in the Introduction, (3.11) reduces to a four-dimensional equation for ψ_{++}, *i.e.*

$$\psi_{++}(p') = \psi^0_{++}(p') + G_{++}(p';W)\int d^4p \ I(p',p)_{++,++}\psi_{++}(p) \ , \tag{3.12}$$

with the Green-function

$$G_{++}(p;W) = \frac{i}{(2\pi)^4}\left[\frac{M}{E(\mathbf{p})}\right]^2 \Lambda_+^a(\mathbf{p})\Lambda_+^b(-\mathbf{p}) \cdot$$
$$\times \frac{1}{[\frac{1}{2}W + p_0 - E(\mathbf{p}) + i\delta]}\frac{1}{[\frac{1}{2}W - p_0 - E(\mathbf{p}) + i\delta]} \ . \tag{3.13}$$

IV. THREE-DIMENSIONAL EQUATION

In this section we apply the procedure of Salpeter [20] and perform the p_0-integrations. Therewith we follow closely the method of Klein [22]. This way we gain a clear definition of the potential. Specifically, the rules for the derivation of the two-pion-exchange potential can be established in this way.

Following [20] we define the three-dimensional wave-function by

$$\phi(\mathbf{p}) = \frac{E(\mathbf{p})}{M}\int_{-\infty}^{\infty}\psi(p_\mu)dp_0 \ . \tag{4.1}$$

where the E/M-factor is introduced to have the non-relativistic normalization for ϕ. The aim of this section is to derive an equation for $\phi(\mathbf{p})$ from the equations for ψ in the foregoing section. This is in general not possible, because we do not know the p_0'-dependence of ψ in these equations. It was shown by Salpeter [20] and by Levy [29], that if one restricts oneself to the ladder approximation and assumes a static potential, then without further approximation a three-dimensional equation holds for $\phi(\mathbf{p})$. Klein [22] improved on this by making the following *ansatz* for the p_0'-dependence of $\psi(p_\mu')$

$$\psi(p_\mu') = \frac{M}{E(\mathbf{p}')}A_W(p_\mu')\phi(\mathbf{p}') \ , \tag{4.2}$$

where

$$A_W(p_\mu') = -\frac{1}{2\pi i}\left\{\frac{1}{\frac{1}{2}W + p_0' - E(\mathbf{p}') + i\delta} + \frac{1}{\frac{1}{2}W - p_0' - E(\mathbf{p}') + i\delta}\right\}$$
$$= -\frac{1}{2\pi i}(W - 2E_{\mathbf{p}'})\left[F_W(\mathbf{p}', p_0')F_W(-\mathbf{p}', -p_0')\right]^{-1} \ . \tag{4.3}$$

We have introduced in the second expression the, frequently used, notation

$$F_W(\mathbf{p}, p_0) = p_0 - E(\mathbf{p}) + \frac{1}{2}W + i\delta \ . \tag{4.4}$$

For dealing with the scattering problem, it is important to notice that this also holds for the free wave-function $\psi^{(0)}(p_\mu')$. Leaving aside for a moment the free particle spinors one has

$$\psi^{(0)}(p_\mu') \propto \delta^4(p' - p_i) = \delta(p_0' - p_{i0})\ \delta^3(\mathbf{p}' - \mathbf{p}_i) = A_W(p')\ \delta^3(\mathbf{p}' - \mathbf{p}_i) \ . \tag{4.5}$$

Using (4.3) for the p_0-dependence in (3.12), one can perform the p_0-integration of both the left- and right-hand side, which leads to a three-dimensional equation for $\phi(\mathbf{p}')$

$$\phi_{++}(\mathbf{p}') = \phi^{(0)}_{++}(\mathbf{p}') + E^{(+)}_2(\mathbf{p}'; W) \int d^3p \, K^{irr}(\mathbf{p}', \mathbf{p}|W) \, \phi_{++}(\mathbf{p}) \ , \tag{4.6}$$

where the Green function is defined as

$$E^{(+)}_2(\mathbf{p}'; W) = \frac{1}{(2\pi)^3} \frac{\Lambda^a_+(\mathbf{p}')\Lambda^b_+(-\mathbf{p}')}{(W - 2E(\mathbf{p}') + i\delta)} \ , \tag{4.7}$$

and where the kernel is given by

$$\begin{aligned} K^{irr}(\mathbf{p}', \mathbf{p}|W) = &-\frac{1}{(2\pi)^2} \frac{M^2}{E(\mathbf{p}')E(\mathbf{p})} (W - 2E(\mathbf{p}'))(W - 2E(\mathbf{p})) \cdot \\ &\times \int_{-\infty}^{+\infty} dp'_0 \int_{-\infty}^{+\infty} dp_0 \left[\{F_W(\mathbf{p}', p'_0) F_W(-\mathbf{p}', -p'_0)\}^{-1} \cdot \right. \\ &\left. \times \ [I(p'_0, \mathbf{p}'; p_0, \mathbf{p})]_{++,++} \{F_W(\mathbf{p}, p_0) F_W(-\mathbf{p}, -p_0)\}^{-1} \right] \ . \end{aligned} \tag{4.8}$$

Notice that (4.6) is known in the literature as the Thompson equation [21].

The M/E-factors in (4.8) are due to the difference between the relativistic and the non-relativistic normalization of the two-particle states. In the following we simply put $M/E(\mathbf{p}) = 1$ in the kernel K (Eq. (4.8)). The corrections to this approximation would give $1/M^2$-corrections to the potentials, which we neglect. Also in Appendix C we neglect the difference in normalization.

The contributions to the two-particle irreducible kernel K^{irr} up to fourth order in the pion-nucleon coupling constant are given by the diagram of Fig. (1) and diagram (b) of Fig. (2). For the definition of the TPE-potential in the Lippmann-Schwinger equation we shall need the complete fourth-order kernel for the Thompson equation (4.6). In operator notation, we get from (4.6)

$$\begin{aligned} \phi_{++} &= \phi^{(0)}_{++} + E^{(+)}_2 \, K^{irr} \, \phi_{++} \\ &= \phi^{(0)}_{++} + E^{(+)}_2 \left(K^{irr} + K^{irr} \, E^{(+)}_2 \, K^{irr} + ... \right) \phi^{(0)}_{++} \\ &\equiv \left(1 + E^{(+)}_2 \, K \right) \phi^{(0)}_{++} \ , \end{aligned} \tag{4.9}$$

which implies for the complete kernel K the integral equation

$$K(\mathbf{p}', \mathbf{p}|W) = K^{irr}(\mathbf{p}', \mathbf{p}|W) + \int d^3p'' \, K^{irr}(\mathbf{p}', \mathbf{p}''|W) \, E^{(+)}_2(\mathbf{p}''; W) \, K(\mathbf{p}'', \mathbf{p}|W) \ . \tag{4.10}$$

Notice that diagram (a) of Fig. (2) is generated from the iterated OPE in (4.10), albeit with the Thompson two-particle propagator (4.7). We calculate the complete kernel $K(\mathbf{p}', \mathbf{p}; W)$ in the next section and define in section VII the potential $V(\mathbf{p}', \mathbf{p}; W)$ such that up to fourth order the Thompson amplitude is recovered completely by using this potential in the Lippmann-Schwinger equation.

V. OPE- AND TPE-KERNELS

In the calculation of the interaction kernel, we restrict ourselves to terms up to and including the fourth order in the pion-nucleon coupling constant. Writing the wave-function as a series in the pion-nucleon coupling, and the interaction kernel as a sum of the second and the fourth order term, we have

$$\begin{aligned} \phi_{++}(\mathbf{p}') &= \phi^{(0)}_{++}(\mathbf{p}') + \phi^{(2)}_{++}(\mathbf{p}') + \phi^{(4)}_{++}(\mathbf{p}') + \dots \ , \\ K(\mathbf{p}', \mathbf{p}|W) &= K^{(2)}(\mathbf{p}', \mathbf{p}|W) + K^{(4)}(\mathbf{p}', \mathbf{p}|W) \ . \end{aligned} \tag{5.1}$$

From (4.10) one sees that, written in operator notation,

$$\begin{aligned} K^{(2)} &= K^{irr(2)} \ , \\ K^{(4)} &= K^{irr(4)} + K^{irr(2)}\ E^{(+)}_2\ K^{irr(2)} \ , \end{aligned} \tag{5.2}$$

and so the $K^{(2)}$-term corresponds to the Feynman diagram in Fig. (1) and the $K^{(4)}$-term to the graphs in Fig. (2). From (4.9) we then find for the wave-function

$$\begin{aligned} \phi^{(2)}_{++} &= E^{(+)}_2 K^{irr(2)}\ \phi^{(0)}_{++} \ , \\ \phi^{(4)}_{++} &= E^{(+)}_2 \left[K^{irr(2)}\ E^{(+)}_2\ K^{irr(2)} + K^{irr(4)}\right]\ \phi^{(0)}_{++} \ . \end{aligned} \tag{5.3}$$

So, the second-order Feynman diagram Fig. (1) gives for the second-order wave-function

$$\begin{aligned} \phi^{(2)}_{++}(\mathbf{p}') = E^{(+)}_2(\mathbf{p}'; W) \int d\mathbf{p} \cdot \Big[&- (2\pi)^{-2}(W - 2E(\mathbf{p}'))(W - 2E(\mathbf{p})) \cdot \\ &\times \int dp'_0 \int dp_0 \int dk_0 \int d^3k\ \delta^4(p - p' - k) \cdot \\ &\times \left[F_W(\mathbf{p}', p'_0) F_W(-\mathbf{p}', -p'_0)\right]^{-1} \cdot \\ &\times \left[k^2 - m^2 + i\delta\right]^{-1} \left\{\Gamma^a_i \Gamma^b_i\right\} \cdot \\ &\times \left[F_W(\mathbf{p}, p_0) F_W(-\mathbf{p}, -p_0)\right]^{-1} \Big]\ \phi^{(0)}_{++}(\mathbf{p}) \ . \end{aligned} \tag{5.4}$$

Using the familiar integral representation

$$\delta(p_0 - p'_0 - k_0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\alpha\ \ \exp i\alpha(p_0 - p'_0 - k_0) \ , \tag{5.5}$$

one finds upon integration over p_0 and p'_0

$$K^{(2)}(\mathbf{p}', \mathbf{p}|W) = -\frac{\sum_i \Gamma^a_i \Gamma^b_i}{\omega_{\mathbf{k}}[E_{\mathbf{p}} + E_{\mathbf{p}'} - W + \omega_{\mathbf{k}}]} \ , \tag{5.6}$$

which corresponds to the two three-dimensional momentum-space diagrams of Fig. (3) (for more details see appendix A).

The fourth order Feynman diagrams, the so-called planar-box and crossed-box diagram, lead to the following expression for the fourth-order wave-function

$$
\begin{aligned}
\phi^{(4)}_{++}(\mathbf{p}') = E_2^{(+)}(\mathbf{p}';W)\ \int d^3p \cdot \Big[& -(2\pi)^{-2}(W-2E(\mathbf{p}'))(W-2E(\mathbf{p}))\cdot \\
& \times \int dp'_0 \int dp_0 \Big\{ i(2\pi)^{-4} \int dk_0 \int dk'_0 \int d^3k \int d^3k'\ \delta^4(p-p'-k-k')\cdot \\
& \times [F_W(\mathbf{p}',p'_0)F_W(-\mathbf{p}',-p'_0)]^{-1}\,[k^2-m^2+i\delta]^{-1}[k'^2-m^2+i\delta]^{-1}\cdot \\
& \times \Big\{[\Gamma_j F_W^{-1}(\mathbf{p}-\mathbf{k},p_0-k_0)\Gamma_i]^{(a)}[\Gamma_j F_W^{-1}(-\mathbf{p}+\mathbf{k},-p_0+k_0)\Gamma_i]^{(b)} \\
& + [\Gamma_j F_W^{-1}(\mathbf{p}-\mathbf{k},p_0-k_0)\Gamma_i]^{(a)}[\Gamma_i F_W^{-1}(-\mathbf{p}'-\mathbf{k},-p'_0-k_0)\Gamma_j]^{(b)}\Big\}\Big\} \\
& \times [F_W(\mathbf{p},p_0)F_W(-\mathbf{p},-p_0)]^{-1}\Big]\ \phi^{(0)}_{++}(\mathbf{p})\ .
\end{aligned} \tag{5.7}
$$

Here Γ denotes the PV-vertex. The expression between the curly brackets is the fourth order contribution to the kernel $I(p';p)_{++,++}$. In the latter we used the two-nucleon Green function (3.13) for the intermediate states, in accordance with the pair-suppression hypothesis. Also we have put here $M/E = 1$. Note that the first term between the curly brackets corresponds to the planar-box diagram and the second term to the crossed-box diagram. The vertex factors Γ_i follow from the interaction Lagrangian (2.4)

$$
\begin{aligned}
\bar{u}(\mathbf{p}')\Gamma^{(a)}u(\mathbf{p}) &= +i\,(f/m_\pi)\,\bar{u}(\mathbf{p}')\gamma_5\boldsymbol{\gamma}\cdot(\mathbf{p}-\mathbf{p}')u(\mathbf{p}) \\
\bar{u}(-\mathbf{p}')\Gamma^{(b)}u(-\mathbf{p}) &= -i\,(f/m_\pi)\,\bar{u}(-\mathbf{p}')\gamma_5\boldsymbol{\gamma}\cdot(\mathbf{p}-\mathbf{p}')u(-\mathbf{p})\ .
\end{aligned} \tag{5.8}
$$

From the explicit expression in (5.7) it is clear that one can perform the integration over the energy variables p'_0, p_0, and k_0.

To do these energy integrations we found it to be very convenient to introduce auxiliary energy variables, using δ-functions. This in order to avoid the occurrence of more than one integration variable in any factor of the denominators. These δ-functions are then treated using the standard integral representation (5.5). For example, in the case of the second meson propagator we introduce in (5.7) the variable $k'_0 = p_0 - p'_0 - k_0$ and use

$$
\delta(k'_0 - p_0 + p'_0 + k_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\alpha\ \exp i\alpha(k'_0 - p_0 + p'_0 + k_0)\ . \tag{5.9}
$$

The details of the integrations are discussed in Appendix A and the results for the planar- and crossed-box diagram are as follows:

(i) <u>The planar-box diagram</u>: We introduce the integral

$$
\begin{aligned}
I_{\|}(\mathbf{p}',\mathbf{p}|W) = (W-2E\mathbf{p}')(W-2E_{\mathbf{p}})\cdot & \int dp'_0 \int dp_0 \int dk_0 \\
& \times [k^2-m^2]^{-1}[(p-p'-k)^2-m^2]^{-1}\cdot \\
& \times [F_W(\mathbf{p}',p'_0)F_W(-\mathbf{p}',-p'_0)]^{-1}\cdot \\
& \times [F_W(\mathbf{p}-\mathbf{k},p_0-k_0)F_W(-\mathbf{p}+\mathbf{k},-p_0+k_0)]^{-1}\cdot \\
& \times [F_W(\mathbf{p},p_0)F_W(-\mathbf{p},-p_0)]^{-1}\ .
\end{aligned} \tag{5.10}
$$

Doing the integrations leads in a straigthforward manner to the terms given in (A5) of Appendix A. The first term is easily seen to correspond to the two three-dimensional momentum space planar BW-diagrams of Fig. (4). The second term comes from the four TMO-diagrams of Fig. (6). We get for the contributions to the interaction kernel

$$
\begin{aligned}
K^{(BW)}_{\|}(\mathbf{p}',\mathbf{p}|W) &= -\frac{2}{(2\pi)^3}\int \frac{d^3k}{4\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}\frac{1}{[E_{\mathbf{p}}+E_{\mathbf{p}'}-W+\omega_{\mathbf{k}}+\omega_{\mathbf{k}'}]}\cdot \\
&\times \frac{[\Gamma_j\Lambda_+(\mathbf{p}-\mathbf{k})\Gamma_i]^a}{[E_{\mathbf{p}}+E_{\mathbf{p}-\mathbf{k}}-W+\omega_{\mathbf{k}}]}\frac{[\Gamma_j\Lambda_+(-\mathbf{p}+\mathbf{k})\Gamma_i]^b}{[E_{\mathbf{p}'}+E_{\mathbf{p}-\mathbf{k}}-W+\omega_{\mathbf{k}'}]}\cdot \\
K^{(TMO)}_{\|}(\mathbf{p}',\mathbf{p}|W) &= -\frac{4}{(2\pi)^3}\int \frac{d^3k}{4\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}\frac{1}{[2E_{\mathbf{p}-\mathbf{k}}-W]}\cdot \\
&\times \frac{[\Gamma_j\Lambda_+(\mathbf{p}-\mathbf{k})\Gamma_i]^a}{[E_{\mathbf{p}}+E_{\mathbf{p}-\mathbf{k}}-W+\omega_{\mathbf{k}}]}\frac{[\Gamma_j\Lambda_+(-\mathbf{p}+\mathbf{k})\Gamma_i]^b}{[E_{\mathbf{p}'}+E_{\mathbf{p}-\mathbf{k}}-W+\omega_{\mathbf{k}'}]}\ ,
\end{aligned}
\tag{5.11}
$$

where $\omega=\sqrt{\mathbf{k}^2+m^2}$ and $\omega'=\sqrt{\mathbf{k}'^2+m^2}$ with $\mathbf{k}'\equiv\mathbf{p}-\mathbf{p}'-\mathbf{k}$.
(ii) <u>The crossed-box diagram</u>: We introduce for the crossed-box diagram the integral

$$
\begin{aligned}
I_X(\mathbf{p}',\mathbf{p}|W) &= (W-2E\mathbf{p}')(W-2E_{\mathbf{p}})\int dp_0'\int dp_0\int dk_0\ \cdot \\
&\times [k^2-m^2]^{-1}[(p-p'-k)^2-m^2]^{-1}\cdot \\
&\times [F_W(\mathbf{p}',p_0')F_W(-\mathbf{p}',-p_0')]^{-1}\cdot \\
&\times [F_W(\mathbf{p}-\mathbf{k},p_0-k_0)F_W(-\mathbf{p}'-\mathbf{k},-p_0'-k_0)]^{-1}\cdot \\
&\times [F_W(\mathbf{p},p_0)F_W(-\mathbf{p},-p_0)]^{-1}\ .
\end{aligned}
\tag{5.12}
$$

Doing these integrations leads after some algebra to the terms given in (A10) of Appendix A. These terms correspond to the six three-dimensional momentum space perturbation diagrams of Fig. (5). The contributions to the interaction kernel are

$$
\begin{aligned}
K^{(5a)}_{X}(\mathbf{p}',\mathbf{p}|W) &= -\frac{1}{(2\pi)^3}\int \frac{d^3k}{4\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}\frac{1}{[E_{\mathbf{p}}+E_{\mathbf{p}'}-W+\omega_{\mathbf{k}}+\omega_{\mathbf{k}'}]}\cdot \\
&\times \frac{[\Gamma_j\Lambda_+(\mathbf{p}-\mathbf{k})\Gamma_i]^a}{[E_{\mathbf{p}-\mathbf{k}}+E_{\mathbf{p}}-W+\omega_{\mathbf{k}}]}\frac{[\Gamma_i\Lambda_+(-\mathbf{p}'-\mathbf{k})\Gamma_j]^b}{[E_{\mathbf{p}'+\mathbf{k}}+E_{\mathbf{p}'}-W+\omega_{\mathbf{k}}]}\ , \\
K^{(5b)}_{X}(\mathbf{p}',\mathbf{p}|W) &= -\frac{1}{(2\pi)^3}\int \frac{d^3k}{4\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}\frac{1}{[E_{\mathbf{p}}+E_{\mathbf{p}'}-W+\omega_{\mathbf{k}}+\omega_{\mathbf{k}'}]}\cdot \\
&\times \frac{[\Gamma_j\Lambda_+(\mathbf{p}-\mathbf{k})\Gamma_i]^a}{[E_{\mathbf{p}'+\mathbf{k}}+E_{\mathbf{p}}-W+\omega_{\mathbf{k}'}]}\frac{[\Gamma_i\Lambda_+(-\mathbf{p}'-\mathbf{k})\Gamma_j]^b}{[E_{\mathbf{p}-\mathbf{k}}+E_{\mathbf{p}'}-W+\omega_{\mathbf{k}'}]}\ ,
\end{aligned}
$$

$$K_X^{(5c)}(\mathbf{p}',\mathbf{p}|W) = -\frac{1}{(2\pi)^3}\int \frac{d^3k}{4\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}\frac{1}{[E_{\mathbf{p}-\mathbf{k}}+E_{\mathbf{p}'+\mathbf{k}}-W+\omega_{\mathbf{k}}+\omega_{\mathbf{k}'}]}\cdot$$
$$\times\frac{[\Gamma_j\Lambda_+(\mathbf{p}-\mathbf{k})\Gamma_i]^a}{[E_{\mathbf{p}-\mathbf{k}}+E_{\mathbf{p}}-W+\omega_{\mathbf{k}}]}\frac{[\Gamma_i\Lambda_+(-\mathbf{p}'-\mathbf{k})\Gamma_j]^b}{[E_{\mathbf{p}'+\mathbf{k}}+E_{\mathbf{p}'}-W+\omega_{\mathbf{k}}]}\ ,$$

$$K_X^{(5d)}(\mathbf{p}',\mathbf{p}|W) = -\frac{1}{2\pi)^3}\int \frac{d^3k}{4\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}\frac{1}{[E_{\mathbf{p}'+\mathbf{k}}+E_{\mathbf{p}-\mathbf{k}}-W+\omega_{\mathbf{k}}+\omega_{\mathbf{k}'}]}\cdot$$
$$\times\frac{[\Gamma_j\Lambda_+(\mathbf{p}-\mathbf{k})\Gamma_i]^a}{[E_{\mathbf{p}-\mathbf{k}}+E_{\mathbf{p}'}-W+\omega_{\mathbf{k}'}]}\frac{[\Gamma_i\Lambda_+(-\mathbf{p}'-\mathbf{k})\Gamma_j]^b}{[E_{\mathbf{p}'+\mathbf{k}}+E_{\mathbf{p}}-W+\omega_{\mathbf{k}'}]}\ ,$$

$$K_X^{(5e)}(\mathbf{p}',\mathbf{p}|W) = -\frac{1}{(2\pi)^3}\int \frac{d^3k}{4\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}\frac{1}{[E_{\mathbf{p}'+\mathbf{k}}+E_{\mathbf{p}-\mathbf{k}}-W+\omega_{\mathbf{k}}+\omega_{\mathbf{k}'}]}\cdot$$
$$\times\frac{[\Gamma_j\Lambda_+(\mathbf{p}-\mathbf{k})\Gamma_i]^a}{[E_{\mathbf{p}-\mathbf{k}}+E_{\mathbf{p}}-W+\omega_{\mathbf{k}}]}\frac{[\Gamma_i\Lambda_+(-\mathbf{p}'-\mathbf{k})\Gamma_j]^b}{[E_{\mathbf{p}-\mathbf{k}}+E_{\mathbf{p}'}-W+\omega_{\mathbf{k}'}]}\ ,$$

$$K_X^{(5f)}(\mathbf{p}',\mathbf{p}|W) = -\frac{1}{(2\pi)^3}\int \frac{d^3k}{4\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}\frac{1}{[E_{\mathbf{p}'+\mathbf{k}}+E_{\mathbf{p}-\mathbf{k}}-W+\omega_{\mathbf{k}}+\omega_{\mathbf{k}'}]}\cdot$$
$$\times\frac{[\Gamma_j\Lambda_+(\mathbf{p}-\mathbf{k})\Gamma_i]^a}{[E_{\mathbf{p}'+\mathbf{k}}+E_{\mathbf{p}}-W+\omega_{\mathbf{k}'}]}\frac{[\Gamma_i\Lambda_+(-\mathbf{p}'-\mathbf{k})\Gamma_j]^b}{[E_{\mathbf{p}'+\mathbf{k}}+E_{\mathbf{p}'}-W+\omega_{\mathbf{k}}]}\ . \tag{5.13}$$

VI. TREATMENT GAUSSIAN FORM FACTOR

The generalization of the interaction kernels, given in section V, to the case with a πNN-form factor is achieved by making in (5.7) the substitution

$$[k^2-m^2+i\delta]^{-1}\longrightarrow\int_0^\infty d\mu^2\ \frac{\rho(\mu^2)}{k^2-\mu^2+i\delta} \tag{6.1}$$

for each OPE propagator. Here at the right-hand side $\rho(\mu^2)$ is the spectral function, representing the form factors involved in OPE.

Like in the soft-core OBE-potential [3], we treat the case with a Gaussian form factor $F(t)$ in the OPE-amplitude. Here $t=(p-p')^2$, the relativistic momentum transfer of the nucleons at the πNN-vertices. From

$$t=(E_{\mathbf{p}}-E_{\mathbf{p}-\mathbf{k}})^2-\mathbf{k}^2=-\mathbf{k}^2+\frac{1}{4M^2}\left(2\mathbf{p}\cdot\mathbf{k}-\mathbf{k}^2\right)^2+O(\frac{1}{M^4})\ , \tag{6.2}$$

where $\mathbf{k}=\mathbf{p}-\mathbf{p}'$, it is seen that one can use in the nucleon form factor $t=-\mathbf{k}^2$ to a very good approximation at low and medium energies. So, putting $t=-\mathbf{k}^2<0$ we have

$$F(\mathbf{k}^2)=e^{-\mathbf{k}^2/\Lambda^2}\ , \tag{6.3}$$

where Λ-denotes the cut-off mass. Using Gaussian form factors, means for $t = k^2 \approx -\mathbf{k}^2 < 0$, *i.e.* for spacelike momentum transfer, that

$$\int_0^\infty d\mu^2 \ \frac{\rho(\mu^2)}{\mathbf{k}^2+\mu^2} \longrightarrow \frac{e^{-\mathbf{k}^2/\Lambda^2}}{\mathbf{k}^2+m^2} \ . \tag{6.4}$$

Notice here that the use of the disperson relation (6.1) as an intermediate step is essential, because for time-like k^2 the Gaussian form (6.4) is not valid in principle. Only after the performance of the energy-integrations we can use (6.4).

To analyse the consequences for TPEP, consider for example graph (a) of Fig. (2). Interchanging the dp_0- etc. integrals with the spectral integrals $d\mu^2$ etc., all expressions for the TPE kernels $K(\mathbf{p}',\mathbf{p}|W)$ in the foregoing section get in front the spectral integrals. These spectral integrals can be treated by application of the substitution rule given in (6.4). To illustrate the procedure in more detail, we consider the parallel graphs of Fig. (4). In the adiabatic approximation, inserting the spectral representation (6.1) for the OPE-amplitude leads intrinsically to the integral

$$\tilde{J}_1 = \int_0^\infty d\mu_1^2 \int_0^\infty d\mu_2^2 \ \rho(\mu_1^2)\rho(\mu_2^2) \ \frac{1}{\omega(\mu_1)\omega(\mu_2)} \frac{1}{[\omega(\mu_1)+\omega(\mu_2)]} \ , \tag{6.5}$$

where $\omega(\mu_1) = \sqrt{\mathbf{k}_1^2+\mu_1^2}$ and $\omega(\mu_2) = \sqrt{\mathbf{k}_2^2+\mu_2^2}$. Here $\mathbf{k}_1$ and $\mathbf{k}_2$ are the three-momenta carried by the two pions in the diagram [36]. It is shown in Appendix B that we can write

$$\tilde{J}_1 = \frac{2}{\pi}\int_0^\infty d\lambda \left[\int_0^\infty d\mu_1^2 \ \frac{\rho(\mu_1^2)}{\mathbf{k}_1^2+\mu_1^2+\lambda^2}\right] \left[\int_0^\infty d\mu_2^2 \ \frac{\rho(\mu_2^2)}{\mathbf{k}_2^2+\mu_2^2+\lambda^2}\right] \ . \tag{6.6}$$

From the substitution rule given in (6.4) one sees that

$$\tilde{J}_1 \simeq \frac{2}{\pi}\int_0^\infty d\lambda \left[\frac{e^{-(\mathbf{k}_1^2+\lambda^2)/\Lambda^2}}{\mathbf{k}_1^2+m^2+\lambda^2}\right] \left[\frac{e^{-(\mathbf{k}_2^2+\lambda^2)/\Lambda^2}}{\mathbf{k}_2^2+m^2+\lambda^2}\right] \ . \tag{6.7}$$

From this discussion one sees that also in the presence of the Gaussian form factor, one can perform the k_0- and k_0'-integration in the Feynman diagrams.
Notice that in (6.7) we have achieved factorization of the $\mathbf{k}_1$- and $\mathbf{k}_2$-dependence under the λ-integral. This facilitates the Fourier transformation to configuration space. For the evaluation of the TPEP in configuration space, we give in Appendix B a dictionary of Fourier transformations.

VII. DEFINITION TPE-POTENTIAL

In this section we define the potential for the Lippmann-Schwinger equation. We start from the Thompson equation (4.6) and write it in operator notation

$$\phi = \phi^{(0)} + E_2^{(+)} \ K^{irr} \ \phi \ . \tag{7.1}$$

The Green function for the Lippmann-Schwinger equation is given by [37]

$$g(\mathbf{p};W) = \frac{1}{(2\pi)^3}\,\Lambda_+^a(\mathbf{p})\Lambda_+^b(-\mathbf{p})\,\frac{M}{\mathbf{p}_i^2-\mathbf{p}^2+i\delta}\ . \tag{7.2}$$

We make now the approximation $E_2^{(+)} = g(\mathbf{p};W)$ and write (7.1) as

$$\phi = \phi^{(0)} + g\ V\ \phi\ . \tag{7.3}$$

Again, the corrections to this approximation are of order $1/M^2$, which we neglect.

Now we want to determine the potential V up to fourth-order in the pion-nucleon coupling, such that to that order the wave-function and the T-matrix are the same. This implies the following for the potential V

$$\begin{aligned} V^{(2)} &= K^{(2)}\ , \\ V^{(4)} &= K^{(4)} - K^{(2)}\ g\ K^{(2)}\ . \end{aligned} \tag{7.4}$$

These equations have to be taken, where the initial and final states are on the energy-shell.

The second order potential $V^{(2)}$ is given by the diagrams of Fig. (3) taken on energy-shell. This is then equivalent to the potential diagram (a) in Fig. (7). The fourth order potential $V^{(4)}$ consists of two parts. The first part is given by the fourth order diagrams in Figs. (4a–4f) and Figs. (5a–5f). The second part comes from diagram Figs. (6a–6f), from which we subtract the once iterated one-pion contribution

$$T^{(4)}_{Born} = V^{(2)}\ g\ V^{(2)} = K^{(2)}\ g\ K^{(2)}\ . \tag{7.5}$$

For (7.3) the transition from Dirac-spinors to Pauli-spinors, is given in Appendix C. There we derive the Lippmann-Schwinger equation

$$\chi(\mathbf{p}) = \chi^{(0)}(\mathbf{p}) + \tilde{g}(\mathbf{p})\int d^3p'\ \mathcal{V}(\mathbf{p},\mathbf{p}')\ \chi(\mathbf{p}') \tag{7.6}$$

for the Pauli-spinor wave-functions $\chi(\mathbf{p})$. The wave-function $\chi(\mathbf{p})$ and the potential $\mathcal{V}(\mathbf{p},\mathbf{p}')$ in the Pauli spinor-space are defined by

$$\begin{aligned} \phi(\mathbf{p}) &= \sum_{s_a,s_b}\chi_{s_a s_b}(\mathbf{p})\ u_a(\mathbf{p},s_a)u_b(-\mathbf{p},s_b) \\ \chi^{(a)\dagger}_{s_a}\chi^{(b)\dagger}_{s_b}\ \mathcal{V}\ \chi^{(a)}_{s'_a}\chi^{(b)}_{s'_b} &= \bar{u}_a(\mathbf{p},s_a)\bar{u}_b(-\mathbf{p},s_b)\ V(\mathbf{p},\mathbf{p}')\ u_a(\mathbf{p}',s'_a)u_b(-\mathbf{p}',s'_b)\ . \end{aligned} \tag{7.7}$$

Like in the derivation of the OBE-potentials [3,37] we make the approximation

$$E(\mathbf{p}) = M + \mathbf{p}^2/2M \tag{7.8}$$

everywhere in the interaction kernels of section VI, which, of course, is fully justified for low energies only. As a consequence of (7.8), we have a similar expansion of the on-shell energy

$$W = 2\sqrt{\mathbf{p}_i^2+M^2} = 2M + \mathbf{p}_i^2/M\ . \tag{7.9}$$

In contrast to these kind of approximations, of course the full $\mathbf{k}^2$-dependence of the form factors is kept throughout the derivation of the TPEP. Notice that the Gaussian form factors suppress the high momentum transfers strongly. This means that the contribution to the potentials from intermediate states which are far off-energy-shell can not be very large.

For the reduction of the TPEP from Dirac-spinor space to Pauli-spinor space, we use (3.6) for the Λ_+-operators, which leads to matrix elements of the vertex operators between positive energy Dirac-spinors. Up to order $1/M$ this gives in Pauli-spinor space, using the expansion (7.8), the vertex operators

$$\bar{u}(\mathbf{p}')\Gamma^{(a)}(\mathbf{p}',\mathbf{p})u(\mathbf{p}) \approx +i\,(f/m_\pi)\left[\boldsymbol{\sigma}_1\cdot\mathbf{k} \mp \frac{\omega}{2M}\boldsymbol{\sigma}_1\cdot(\mathbf{p}'+\mathbf{p})\right]$$
$$\bar{u}(-\mathbf{p}')\Gamma^{(b)}(\mathbf{p}',\mathbf{p})u(-\mathbf{p}) \approx -i\,(f/m_\pi)\left[\boldsymbol{\sigma}_2\cdot\mathbf{k} \mp \frac{\omega}{2M}\boldsymbol{\sigma}_2\cdot(\mathbf{p}'+\mathbf{p})\right]\ , \tag{7.10}$$

where always $\mathbf{k} \equiv \mathbf{p}-\mathbf{p}' = \mathbf{k}_1+\mathbf{k}_2$. For the Γ-matrix-elements in (7.10), the upper sign applies for the creation and the lower sign for the absorption of the pion at the vertex.

In order to obtain all contributions to the potentials up to order $1/M$, we develop the energy denominators in the expressions for the planar- and the crossed-box diagram (see section VI). For the BW-graphs it is sufficient to keep here the terms up to order $1/M$, however for the TMO-graphs we must also keep the terms of order $(1/M)^2$. Then, we get for example

$$\frac{1}{E_{\mathbf{p}}+E_{\mathbf{p}-\mathbf{k}}-W+\omega} = \frac{1}{\omega}\left[1-\frac{\mathbf{p}^2+(\mathbf{p}-\mathbf{k})^2-2\mathbf{p}_i^2}{2M\omega}\right.$$
$$\left.+\left(\frac{\mathbf{p}^2+(\mathbf{p}-\mathbf{k})^2-2\mathbf{p}_i^2}{2M\omega}\right)^2\right]\ . \tag{7.11}$$

The leading terms, order $O(1)$, give the so-called 'adiabatic' approximation [29]. The next to leading contributions, order $O(1/M)$, are referred to as 'non-adiabatic'.

In the non-adiabatic terms one can easily identify the purely off-energy-shell combinations $\mathbf{p}^2-\mathbf{p}_i^2$, $\mathbf{p}'^2-\mathbf{p}_i^2$, and $\mathbf{p}^2+\mathbf{p}'^2-2\mathbf{p}_i^2$. Henceforth we neglect these off-shell contributions. This means, for example, that $\mathbf{p}^2+(\mathbf{p}-\mathbf{k})^2-2\mathbf{p}_i^2 \approx -2\mathbf{p}\cdot\mathbf{k}+\mathbf{k}^2$, etc.

For the definition of the Fourier transformations to configuration space we introduce the standard vectors

$$\mathbf{k} = \mathbf{p}-\mathbf{p}'\ ,\ \ \mathbf{q} = \frac{1}{2}(\mathbf{p}+\mathbf{p}')\ . \tag{7.12}$$

Like (7.11) we can occasionally exploit the relation

$$\mathbf{k} = \mathbf{p}-\mathbf{p}' = \mathbf{k}_1+\mathbf{k}_2 \tag{7.13}$$

before doing the Fourier transformations.

The second order potential, *i.e.* the OPE-potential, can now readily be written down. For the diagrams of Fig. (3) one finds

$$K^{(2)} = -(f/m_\pi)^2(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)(\boldsymbol{\sigma}_1\cdot\mathbf{k}\ \boldsymbol{\sigma}_2\cdot\mathbf{k})\frac{F(\mathbf{k}^2)}{\omega_{\mathbf{k}}[E_{\mathbf{p}}+E_{\mathbf{p}'}-W+\omega_{\mathbf{k}}]}\ . \tag{7.14}$$

From the expansions in (7.11) one sees that the non-adiabatic contribution vanishes on the energy-shell. So, we find the second order potential

$$\mathcal{V}^{(2)} = -(f/m_\pi)^2(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)(\boldsymbol{\sigma}_1 \cdot \mathbf{k}\ \boldsymbol{\sigma}_2 \cdot \mathbf{k})\frac{F(\mathbf{k}^2)}{\mathbf{k}^2 + m^2} \ . \tag{7.15}$$

Notice that in this case there are no contributions of order $1/M$ from the vertices. The term of second order in $1/M$

$$(f/m_\pi)^2(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)(\boldsymbol{\sigma}_1 \cdot \mathbf{q}\ \boldsymbol{\sigma}_2 \cdot \mathbf{q})F(\mathbf{k}^2)/M^2 \tag{7.16}$$

gives (local and non-local) short range Gaussian potentials of order $(m/M)^2(f^2/4\pi)$. If one neglects the off-energy-shell terms, *i.e.* $\mathbf{q} \cdot \mathbf{k} \approx 0$, there are no $1/M^2$-corrections in (7.10) and so no further $1/M^2$-terms in OPE.

In the following sections we calculate the TPEP in configuration space. In momentum space the TPEP will be of the general form

$$(\mathbf{p}'|V^{(TPE)}|\mathbf{p}) = \int\int d^3k_1 d^3k_2\ \delta^3(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)\ \widetilde{V}(\mathbf{p}', \mathbf{p}; \mathbf{k}_1, \mathbf{k}_2) \ . \tag{7.17}$$

The Fourier transformation reads

$$\begin{aligned} V(r) &= (2\pi)^{-3}\int d^3k\ e^{i\mathbf{k}\cdot\mathbf{r}}(\mathbf{p}'|V^{(TPE)}|\mathbf{p}) \\ &= (2\pi)^{-3}\int\int d^3k_1 d^3k_2\ e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}}\ \widetilde{V}(\mathbf{p}', \mathbf{p}; \mathbf{k}_1, \mathbf{k}_2) \ . \end{aligned} \tag{7.18}$$

We calculate the TPE-contribution to V_C, V_σ, V_T, and V_{SO} respectively the central, the spin-spin, the tensor, and the spin-orbit potential.

VIII. TWO-PION-EXCHANGE POTENTIAL (BW-GRAPHS)

(i) *Adiabatic contributions:*

1. Diagrams (a) and (b) of Fig. (4), correspond to the expression $K_{\parallel}^{(BW)}$ in (5.11), and give

$$\begin{aligned} V_{\parallel}^{(a+b)}(r) = &-(3 - 2\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)\left(\frac{f}{m_\pi}\right)^4 (2\pi)^{-6}\int\int d^3k_1 d^3k_2\ e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}} \\ &\times F(\mathbf{k}_1^2)F(\mathbf{k}_2^2)\ \frac{(a - i\ \mathbf{b}\cdot\boldsymbol{\sigma}_1)(a - i\ \mathbf{b}\cdot\boldsymbol{\sigma}_2)}{2\omega^2(\mathbf{k}_1)\omega^2(\mathbf{k}_2)[\omega(\mathbf{k}_1) + \omega(\mathbf{k}_2)]} \ , \end{aligned} \tag{8.1}$$

where

$$a = \mathbf{k}_1 \cdot \mathbf{k}_2 \ \ , \ \ \mathbf{b} = \mathbf{k}_1 \times \mathbf{k}_2 \ . \tag{8.2}$$

2. Diagram (a) and (c) of Fig. (5), correspond to the expressions $K_X^{(5a)}$ and $K_X^{(5c)}$ in (5.13), and give

$$V_X^{(a+c)}(r) = -(3+2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{f}{m_\pi}\right)^4\ (2\pi)^{-6}\int\int d^3k_1 d^3k_2\ e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}}$$

$$\times\ F(\mathbf{k}_1^2)F(\mathbf{k}_2^2)\ \frac{(a-i\mathbf{b}\cdot\boldsymbol{\sigma}_1)(a+i\mathbf{b}\cdot\boldsymbol{\sigma}_2)}{2\omega^3(\mathbf{k}_1)\omega(\mathbf{k}_2)[\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)]}\ . \tag{8.3}$$

3. Diagram (b) and (d) of Fig. (5), correspond to the expressions $K_X^{(5b)}$ and $K_X^{(5d)}$ in (5.13), and give

$$V_X^{(b+d)}(r) = -(3+2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{f}{m_\pi}\right)^4\ (2\pi)^{-6}\int\int d^3k_1 d^3k_2\ e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}}$$

$$\times\ F(\mathbf{k}_1^2)F(\mathbf{k}_2^2)\ \frac{(a-i\mathbf{b}\cdot\boldsymbol{\sigma}_1)(a+i\mathbf{b}\cdot\boldsymbol{\sigma}_2)}{2\omega(\mathbf{k}_1)\omega^3(\mathbf{k}_2)[\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)]}\ . \tag{8.4}$$

4. Diagram (e) and (f) of Fig. (5), correspond to the expressions $K_X^{(5e)}$ and $K_X^{(5f)}$ in (5.13), and give

$$V_X^{(e+f)}(r) = -(3+2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{f}{m_\pi}\right)^4\cdot(2\pi)^{-6}\int\int d^3k_1 d^3k_2\ e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}}$$

$$\times\ F(\mathbf{k}_1^2)F(\mathbf{k}_2^2)\ \frac{(a-i\mathbf{b}\cdot\boldsymbol{\sigma}_1)(a+i\mathbf{b}\cdot\boldsymbol{\sigma}_2)}{2\omega^2(\mathbf{k}_1)\omega^2(\mathbf{k}_2)[\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)]}\ . \tag{8.5}$$

These diagrams were calculated by Brueckner and Watson for point couplings (see [25], Eq. (60)). Using the fact that terms with $(\mathbf{k}_1\cdot\mathbf{k}_2)[\mathbf{k}_1\times\mathbf{k}_2\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)]$ in the integrand vanish, we find for these diagrams

$$\begin{aligned} V_{BW}^{(1)}(r) = &-\frac{1}{(2\pi)^6}\left(\frac{f}{m_\pi}\right)^4\int\int d^3k_1 d^3k_2\ e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}}\ \frac{e^{-\mathbf{k}_1^2/\Lambda^2}e^{-\mathbf{k}_2^2/\Lambda^2}}{\omega^3(\mathbf{k}_1)\omega(\mathbf{k}_2)}\cdot \\ &\times\Bigg\{\left[\frac{3}{\omega(\mathbf{k}_2)}+\frac{2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2}{\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)}\right](\mathbf{k}_1\cdot\mathbf{k}_2)^2+ \\ &+\left[\frac{3}{\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)}+\frac{2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2}{\omega(\mathbf{k}_2)}\right](\mathbf{k}_1\times\mathbf{k}_2\cdot\boldsymbol{\sigma}_1)(\mathbf{k}_1\times\mathbf{k}_2\cdot\boldsymbol{\sigma}_2)\Bigg\}\cdot \end{aligned} \tag{8.6}$$

The evaluation of the momentum integrations is readily performed by using the definitions and formulas given in Appendices B and D. This leads to the potentials:

$$\begin{aligned} V_C^{(1)}(BW) = &-\left(\frac{f}{m_\pi}\right)^4\frac{2}{\pi}\int_0^\infty\frac{d\lambda}{\lambda^2}\Bigg\{3\left[\frac{2}{r^2}F'(r)I'(r)+F''(r)I''(r)\right]+ \\ &+2(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left[\frac{2}{r^2}F'(r)G'(r)+F''(r)G''(r)\right]\Bigg\}\ , \end{aligned}$$

$$
\begin{aligned}
V_\sigma^{(1)}(BW) = &-\left(\frac{f}{m_\pi}\right)^4 \frac{2}{\pi}\int_0^\infty \frac{d\lambda}{\lambda^2}\left\{ 2\left[\frac{1}{r^2}F'(r)G'(r)+\frac{1}{r}F''(r)G'(r)+\right.\right.\\
&+\left.\frac{1}{r}F'(r)G''(r)\right]+\frac{4}{3}(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left[\frac{1}{r^2}F'(r)I'(r)+\right.\\
&+\left.\left.\frac{1}{r}F''(r)I'(r)+\frac{1}{r}F'(r)I''(r)\right]\right\} ,
\end{aligned}
$$

$$
\begin{aligned}
V_T^{(1)}(BW) = &-\left(\frac{f}{m_\pi}\right)^4 \frac{2}{\pi}\int_0^\infty \frac{d\lambda}{\lambda^2}\left\{\left[\left(\frac{1}{r}F'(r)-F''(r)\right)\frac{1}{r}G'(r)+\right.\right.\\
&+\left.\frac{1}{r}F'(r)\left(\frac{1}{r}G'(r)-G''(r)\right)\right]+\frac{2}{3}(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\cdot\\
&\times\left.\left[\left(\frac{1}{r}F'(r)-F''(r)\right)\frac{1}{r}I'(r)+\frac{1}{r}F'(r)\left(\frac{1}{r}I'(r)-I''(r)\right)\right]\right\} ,
\end{aligned}
\tag{8.7}
$$

with

$$
\begin{aligned}
I(r) &= I_2(m,r) ,\\
F(r) &= I_2(m,r) - I_2(\sqrt{m^2+\lambda^2},r)\exp\left(-\lambda^2/\Lambda^2\right) ,\\
G(r) &= I_2(\sqrt{m^2+\lambda^2},r)\exp\left(-\lambda^2/\Lambda^2\right) ,
\end{aligned}
\tag{8.8}
$$

where $I_2(m,r)$ is defined in Appendix B, Eq. (B3). Above we introduced the notations $F'(r)\equiv dF/dr$ and $F''(r)\equiv d^2F/dr^2$ etc. Henceforth we employ this notation throughout the following.

(ii) *Adiabatic contributions,* $1/M$*-terms:*
Here we give the $1/M$-contributions ΔV from the $1/M$-terms in the pion-nucleon vertices for the diagrams of Fig. (5). Similar $1/M$-contributions from the diagrams of Fig. (4) cancel each other.
1. Diagram (a) and (b) of Fig. (5) give

$$
\begin{aligned}
\Delta V_X^{a-b}(r) = &-(3+2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{1}{M}\right)\left(\frac{f}{m_\pi}\right)^4(2\pi)^{-6}\int\!\!\int d^3k_1 d^3k_2\ e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}}\\
&\times F(\mathbf{k}_1^2)F(\mathbf{k}_2^2)\frac{(\omega_1-\omega_2)}{4\omega_1^3\omega_2^3}\left[\omega_1\left\{-\mathbf{k}_2^2(\mathbf{k}_1\cdot\mathbf{k}_2)+\right.\right.\\
&+i(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\left[(\mathbf{k}_1\cdot\mathbf{k}_2)(\mathbf{Q}\times\mathbf{k}_2)-(\mathbf{Q}\cdot\mathbf{k}_2)(\mathbf{k}_1\times\mathbf{k}_2)\right]+\\
&+\left.\left[\boldsymbol{\sigma}_1\cdot(\mathbf{k}_1\times\mathbf{k}_2)\boldsymbol{\sigma}_2\cdot(\mathbf{Q}\times\mathbf{k}_2)-\boldsymbol{\sigma}_1\cdot(\mathbf{Q}\times\mathbf{k}_2)\boldsymbol{\sigma}_2\cdot(\mathbf{k}_1\times\mathbf{k}_2)\right]\right\}\\
&\left.-\omega_2\left\{\ldots\right\}_{\mathbf{k}_1\Longleftrightarrow\mathbf{k}_2}\right] .
\end{aligned}
\tag{8.9}
$$

2. Diagram (c), (d), (e) and (f) of Fig. (5) give

$$\Delta V_X^{c-f}(r) = -(3+2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{1}{M}\right)\left(\frac{f}{m_\pi}\right)^4 (2\pi)^{-6}\int\!\!\int d^3k_1 d^3k_2\; e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}}$$

$$\times\, F(\mathbf{k}_1^2)F(\mathbf{k}_2^2)\;\frac{(\omega_1+\omega_2)}{4\omega_1^3\omega_2^3}\Big[\,\omega_1 - \mathbf{k}_2^2(\mathbf{k}_1\cdot\mathbf{k}_2) +$$

$$+i(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\left[(\mathbf{k}_1\cdot\mathbf{k}_2)(\mathbf{Q}\times\mathbf{k}_2) - (\mathbf{Q}\cdot\mathbf{k}_2)(\mathbf{k}_1\times\mathbf{k}_2)\right] +$$

$$+\left[\boldsymbol{\sigma}_1\cdot(\mathbf{k}_1\times\mathbf{k}_2)\boldsymbol{\sigma}_2\cdot(\mathbf{Q}\times\mathbf{k}_2) - \boldsymbol{\sigma}_2\cdot(\mathbf{k}_1\times\mathbf{k}_2)\boldsymbol{\sigma}_1\cdot(\mathbf{Q}\times\mathbf{k}_2)\right]\Big\}$$

$$+\omega_2\Big\{\ldots\Big\}_{\mathbf{k}_1\Longleftrightarrow\mathbf{k}_2}\Big]\;. \tag{8.10}$$

Taking together the contributions from diagrams (a)–(f), we find

$$\Delta V_X^{a-f}(r) = (3+2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{1}{2M}\right)\left(\frac{f}{m_\pi}\right)^4 (2\pi)^{-6}\int\!\!\int d^3k_1 d^3k_2\; e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}}$$

$$\times\, F(\mathbf{k}_1^2)F(\mathbf{k}_2^2)\;\frac{1}{\omega_1^2\omega_2^2}\Big[\,(\mathbf{k}_1\cdot\mathbf{k}_2)(\mathbf{k}_1^2+\mathbf{k}_2^2)$$

$$-i(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\left[(\mathbf{k}_1\cdot\mathbf{k}_2)\mathbf{Q}\times(\mathbf{k}_1+\mathbf{k}_2) + \mathbf{Q}\cdot(\mathbf{k}_1-\mathbf{k}_2)(\mathbf{k}_1\times\mathbf{k}_2)\right] +$$

$$-\left[\boldsymbol{\sigma}_1\cdot(\mathbf{k}_1\times\mathbf{k}_2)\boldsymbol{\sigma}_2\cdot\mathbf{Q}\times(\mathbf{k}_1+\mathbf{k}_2)) - \boldsymbol{\sigma}_2\cdot(\mathbf{k}_1\times\mathbf{k}_2)\boldsymbol{\sigma}_1\cdot\mathbf{Q}\times(\mathbf{k}_1+\mathbf{k}_2)\right]\Big]\;. \tag{8.11}$$

Using again the integrals given in Appendix B and (D5), we find the following contribution to the potentials:

$$V_C^{(2)}(BW) = (3+2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{1}{M}\right)\left(\frac{f}{m_\pi}\right)^4\left[-\frac{2}{r^2}I_2'(r)+\frac{2}{r}I_2''(r)+I_2'''(r)\right]\;I_2'(r)\;,$$

$$V_{SO}^{(2)}(BW) = (3+2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{2}{M}\right)\left(\frac{f}{m_\pi}\right)^4\left[\frac{1}{r}I_2'(r)+I_2''(r)\right]\;\frac{1}{r}I_2'(r)\;. \tag{8.12}$$

(iii) *Non-adiabatic contributions:*
Here we give the $1/M$-contributions ΔV from the $1/M$-terms due to the non-adiabatic expansion of the energy denominators in the intermediate states. In this section we calculate only the diagrams (a) and (b) of Fig. (4) and all the diagrams of Fig. (5). Similar $1/M$- and $1/M^2$-contributions from the diagrams (c), (d), (e), and (f) of Fig. (6) are worked out in the next section.
1. Diagram (a) and (b) of Fig. (4) give

$$\Delta V_\parallel^{a-b}(r) = -(3-2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{1}{M}\right)\left(\frac{f}{m_\pi}\right)^4 (2\pi)^{-6}\int\!\!\int d^3k_1 d^3k_2\; e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}}$$

$$\times \frac{1}{4\omega_1^3\omega_2^3}(\mathbf{k}_1\cdot\mathbf{k}_2)\,(a-i\ \mathbf{b}\cdot\boldsymbol{\sigma}_1)\,(a-i\ \mathbf{b}\cdot\boldsymbol{\sigma}_2)\,F(\mathbf{k}_1^2)F(\mathbf{k}_2^2)\ . \tag{8.13}$$

It appears that this contribution from the BW-graphs cancels exactly against a similar contribution from the TMO-graphs (see next section). Hence, we have not worked out this term further in more detail.
2. Diagrams (a)–(f) of Fig. (5) give

$$\Delta V_X(r) = -(3+2\ \boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{1}{M}\right)\left(\frac{f}{m_\pi}\right)^4\cdot(2\pi)^{-6}\int\!\!\int d^3k_1 d^3k_2\ e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}}$$
$$\times \frac{(\omega_1^2+\omega_2^2)}{2\omega_1^4\omega_2^4}(\mathbf{k}_1\cdot\mathbf{k}_2)\,(a-i\ \mathbf{b}\cdot\boldsymbol{\sigma}_1)\,(a+i\ \mathbf{b}\cdot\boldsymbol{\sigma}_2)\,F(\mathbf{k}_1^2)F(\mathbf{k}_2^2) \tag{8.14}$$

With the results of the Appendices B and D, we get

$$\begin{aligned}
V_C^{(3)}(BW) =&\quad (3+2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{1}{M}\right)\left(\frac{f}{m_\pi}\right)^4\left[\frac{6}{r^2}\left(\frac{1}{r}I_2'-I_2''\right)\cdot\right.\\
&\left.\times\left(\frac{1}{r}I_4'-I_4''\right)+I_2'''I_4'''\right]\ ,\\
V_\sigma^{(3)}(BW) =& -(3+2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{1}{M}\right)\left(\frac{f}{m_\pi}\right)^4\frac{2}{3}\left[\frac{1}{r^2}\left(\frac{1}{r}I_2'-I_2''\right)\cdot\right.\\
&\left.\times\left(\frac{1}{r}I_4'-I_4''\right)+\left(\frac{1}{r}I_2'-I_2''\right)\frac{1}{r}I_4'''+\left(\frac{1}{r}I_4'-I_4''\right)\frac{1}{r}I_2'''\right]\ ,\\
V_T^{(3)}(BW) =&\quad (3+2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{1}{M}\right)\left(\frac{f}{m_\pi}\right)^4\frac{1}{3}\left[\frac{4}{r^2}\left(\frac{1}{r}I_2'-I_2''\right)\cdot\right.\\
&\left.\times\left(\frac{1}{r}I_4'-I_4''\right)+\left(\frac{1}{r}I_2'-I_2''\right)\frac{1}{r}I_4'''+\left(\frac{1}{r}I_4'-I_4''\right)\frac{1}{r}I_2'''\right]\ .
\end{aligned} \tag{8.15}$$

where $I_4(r) = I_4(m,r)$ is defined in Appendix B, Eq. (B9), etc.

IX. TWO-PION-EXCHANGE POTENTIAL (TMO-GRAPHS)

The contribution from diagrams (a-d) of Fig. (6) corresponds to $K_\parallel^{(TMO)}$ in (5.11), and give

$$K_\parallel^{(TMO)} = -\frac{(3-2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)}{(2\pi)^6}\left(\frac{f}{m_\pi}\right)^4\int\!\!\int d^3k_1 d^3k_2\ F(\mathbf{k}_1^2)F(\mathbf{k}_2^2)\ \frac{e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}}}{\omega(\mathbf{k}_1)\omega(\mathbf{k}_2)}\cdot$$
$$\times\ (a-i\ \mathbf{b}\cdot\boldsymbol{\sigma}_1)(a-i\ \mathbf{b}\cdot\boldsymbol{\sigma}_2)[E_{\mathbf{p}}+E_{\mathbf{p}-\mathbf{k}_1}-W+\omega(\mathbf{k}_1)]^{-1}\cdot$$
$$\times\ [2E_{\mathbf{p}-\mathbf{k}_1}-W]^{-1}[E_{\mathbf{p}'}+E_{\mathbf{p}-\mathbf{k}_1}-W+\omega(\mathbf{k}_2)]^{-1}\ , \tag{9.1}$$

where we have made the approximation $W = 2E_{\mathbf{p}}$.
In order to avoid double counting when solving the Schrödinger equation, we subtract from (9.1) the once iterated OPE (see Fig. (7) diagram (b)),

$$T^{(4)}_{Born} = -\frac{(3-2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)}{(2\pi)^6}\left(\frac{f}{m_\pi}\right)^4 \int\int d^3k_1 d^3k_2\ F(\mathbf{k}_1^2)F(\mathbf{k}_2^2)\ \frac{e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}}}{\omega^2(\mathbf{k}_1)\omega^2(\mathbf{k}_2)}\cdot$$

$$\times\ (a - i\mathbf{b}\cdot\boldsymbol{\sigma}_1)(a - i\mathbf{b}\cdot\boldsymbol{\sigma}_2)[2E_{\mathbf{p}-\mathbf{k}_1} - W]^{-1}\ . \tag{9.2}$$

The remaining difference

$$V_{TMO} \equiv K^{(TMO)}_{\|} - T^{(4)}_{Born} \tag{9.3}$$

is referred to as the TMO-potential. This contribution to the potential is neglected in the BW-potential, but is part of the TMO-potential [26]. Since we take all corrections up to order $1/M$ in the potentials into account, which come from the recoil-corrections in the denominators from the intermediate states, it is clear that this contribution should be included. We evaluate the TMO-contributions up to $1/M$-terms in the potentials.

We notice that the adiabatic contributions vanish. This is obvious for the contribution from the leading terms in all vertices. However, also for the $1/M$-contributions from the vertices one can readily see from the graphs in Fig. (6) and the rules for the vertices, that all these contributions cancel. The non-adiabatic contributions give potentials of zero and first order in $1/M$, and are given below.
(i) *Non-adiabatic contributions,* $1/M$*-terms*:
Expanding the energies in the intermediate states to order $1/M$, we find

$$V^{(1)}_{TMO}(r) = (3-2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{f}{m_\pi}\right)^4\ (2\pi)^{-6}\int\int d^3k_1 d^3k_2\ \frac{e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}}}{\omega^3(\mathbf{k}_1)\omega^2(\mathbf{k}_2)}\cdot$$

$$\times\ F(\mathbf{k}_1^2)F(\mathbf{k}_2^2)\left[a^2 - (\mathbf{b}\cdot\boldsymbol{\sigma}_1)(\mathbf{b}\cdot\boldsymbol{\sigma}_2)\right]\ . \tag{9.4}$$

Using the integrals in Appendix B and (D5) leads to potentials:

$$V^{(1)}_C(TMO) = \quad (3-2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{f}{m_\pi}\right)^4\left\{\frac{2}{r^2}I_2'(r)I_3'(r) + I_2''(r)I_3''(r)\right\}\ ,$$

$$V^{(1)}_\sigma(TMO) = -(3-2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{f}{m_\pi}\right)^4\frac{2}{3}\left\{\frac{1}{r^2}I_2'(r)I_3'(r) + \right.$$

$$\left. + \frac{1}{r}[I_2''(r)I_3'(r) + I_2'(r)I_3''(r)]\right\}\ , \tag{9.5}$$

$$V^{(1)}_T(TMO) = -(3-2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{f}{m_\pi}\right)^4\frac{1}{3}\left\{\frac{1}{r}I_2'(r)\left(\frac{1}{r}I_3'(r) - I_3''(r)\right) + \right.$$

$$\left. + \left(\frac{1}{r}I_2'(r) - I_2''(r)\right)\frac{1}{r}I_3'(r)\right\}\ .$$

where $I_2(r) = I_2(m,r)$ and $I_3(r) = I_3(m,r)$ are given in Appendix B, Eq. (B3) and Eq. (B7).
(ii) *Non-adiabatic contributions, $(1/M)^2$-terms*:
Expanding the energies in the intermediate states to order $(1/M)^2$, we find

$$V^{(2)}_{TMO}(r) = \frac{(3-2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)}{(2\pi)^6}\left(\frac{1}{4M}\right)\left(\frac{f}{m_\pi}\right)^4 \int\int d^3k_1 d^3k_2\ e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}}$$

$$\times F(\mathbf{k}_1^2)F(\mathbf{k}_2^2)\left[a^2 - (\mathbf{b}\cdot\boldsymbol{\sigma}_1)(\mathbf{b}\cdot\boldsymbol{\sigma}_2)\right]\cdot(\mathbf{k}_1\cdot\mathbf{k}_2)\cdot$$

$$\times\left\{\frac{1}{\omega^4(\mathbf{k}_1)\omega^2(\mathbf{k}_2)} + \frac{1}{\omega^2(\mathbf{k}_1)\omega^4(\mathbf{k}_2)} + \frac{1}{\omega^3(\mathbf{k}_1)\omega^3(\mathbf{k}_2)}\right\}\ . \tag{9.6}$$

Notice here the well-known cancellation between the non-adiabatic contribution from the BW-graphs as given in (8.13) and the third term in the curly brackets of (9.6). In the following we therefore ignore this third term and give only the potentials from the first and second term.

Now, using the integrals given in Appendix B, with (D5) and (D5), we find the potentials:

$$V^{(2)}_C(TMO) = -(3-2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{1}{2M}\right)\left(\frac{f}{m_\pi}\right)^4\left\{\frac{6}{r^2}\left(\frac{1}{r}I_2' - I_2''\right)\cdot\right.$$

$$\left.\times\left(\frac{1}{r}I_4' - I_4''\right) + I_2'''I_4'''\right\}\ ,$$

$$V^{(2)}_\sigma(TMO) = -(3-2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{1}{2M}\right)\left(\frac{f}{m_\pi}\right)^4\frac{2}{3}\left\{\frac{1}{r^2}\left(\frac{1}{r}I_2' - I_2''\right)\cdot\right.$$

$$\left.\times\left(\frac{1}{r}I_4' - I_4''\right) + \left(\frac{1}{r}I_2' - I_2''\right)\frac{1}{r}I_4''' + \frac{1}{r}I_2'''\left(\frac{1}{r}I_4' - I_4''\right)\right\}\ , \tag{9.7}$$

$$V^{(2)}_T(TMO) = (3-2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{1}{2M}\right)\left(\frac{f}{m_\pi}\right)^4\frac{1}{3}\left\{\frac{4}{r^2}\left(\frac{1}{r}I_2' - I_2''\right)\cdot\right.$$

$$\left.\times\left(\frac{1}{r}I_4' - I_4''\right) + \left(\frac{1}{r}I_2' - I_2''\right)\frac{1}{r}I_4''' + \frac{1}{r}I_2'''\left(\frac{1}{r}I_4' - I_4''\right)\right\}\ .$$

where $I_2 = I_2(m,r)$ and $I_4 = I_4(m,r)$ are defined in Appendix B, equations (B3) and (B9).

X. RESULTS AND DISCUSSION

The complete TPEP can be written as

$$V_i(TPE) = V_i(BW) + V_i(TMO)\ , \tag{10.1}$$

where

$$V_i(BW) = V_i^{(1)}(BW) + V_i^{(2)}(BW) + V_i^{(3)}(BW) \ , \tag{10.2}$$

$$V_i(TMO) = V_i^{(1)}(TMO) + V_i^{(2)}(TMO) \ , \tag{10.3}$$

with $i = C, \sigma, T$, or SO. The potentials $V_i^\alpha(BW)$ for $\alpha = (1, 2, 3)$ are given in (8.7), (8.12), and (8.15). The potentials $V_i^\alpha(TMO)$ for $\alpha = (1, 2)$ are given in (9.5) and (9.7).

In Figs. (8-11) the contribution from the BW- and the TMO-graphs is shown separately and together (TPE) for $I = 0$ and $I = 1$. In these numerical results we have evaluated the TPEP for $f^2/4\pi = 0.08$ and $\Lambda = 664.52$ MeV. For the spin-orbit potential there are only contributions from the BW-graphs. It is seen that the contribution of the TMO-graphs is in general rather significant. For example for $I = 0$ the sign of the tensor potential is changed as compared to the BW-graph contributions. Of course, all potentials are finite at $r = 0$, and due to the Gaussian form factors they are rather soft. Notice that the non-adiabatic terms contribute rather significantly to the TPE-potentials.

In Figs. (12-15) the tail of the TPEP is compared with that of the one-pion-exchange-potential (OPEP) and the one-boson-exchange-potential (HBEP). The latter contains the contribution from all bosons except for the pseudo-scalars and is taken from [3]. For $I = 1$ both for the tensor and the spin-orbit the TPE-contribution might be helpful to improve the results of the soft-core OBE-model [3] when compared to the recent phase shift analysis of [38].

As compared to the TPEP's in the literature, our potential resembles mostly that of Sugawara and Okubo [27]. This is seen most clearly from the point-coupling limits, which are given in Appendix E. However there are differences due to the fact that, in contrast to [27], we do not apply any wave-function transformations. One reason for this is that we do not want the TPEP to be dependent on another part of the potential like OPEP. More important is that we do not want any interdependence between OPEP and HBEP. This would in principle arise if one applies a transformation to the wave-function which is related to OPE. The reason for the resemblance with [27] is of course that we suppress all contributions from the negative energy states (pair-terms). In this respect this work and that of Sugawara and Okubo [27] is different from, for example, Iddings and Platzman [28] and Partovi and Lomon [39]. Comparing the TPEP's of this work with [39], it appears that the potential tails are all similar. This except for the central potential in the $I = 0$-states, where we find repulsion and [39] has attraction. Although we have no pair terms, the tails of the spin-orbit potentials look similar to those of [28].

Finally, we note the changes that have to be made in our results if one chooses the PS-coupling with pair-suppression. In that case the only change is that the $V_i^{(2)}(BW)$-potentials are absent. They arise from the $1/M$-terms in the pion-nucleon vertex (see 7.10).

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APPENDIX A

We discuss here the treatment of the energy integrals that occur in the evaluation of the TPEP. For that purpose, it is convenient to introduce the following notations

$$\begin{aligned}
\omega &= \omega_{\mathbf{k}} \qquad\qquad , \quad \omega' = \omega_{\mathbf{p}-\mathbf{k}-\mathbf{p}'} \\
A_p &= E_{\mathbf{p}} - \tfrac{1}{2}W \quad , \quad A_{p'} = E_{\mathbf{p}'} - \tfrac{1}{2}W \\
A_{p''} &= E_{\mathbf{p}-\mathbf{k}} - \tfrac{1}{2}W \quad , \quad A_{p'''} = E_{\mathbf{p}'+\mathbf{k}} - \tfrac{1}{2}W
\end{aligned} \tag{A1}$$

where $\omega = \sqrt{\mathbf{k}^2 + m^2}$, $\omega' = \sqrt{\mathbf{k}'^2 + m^2}$, and $\mathbf{k}' \equiv \mathbf{p} - \mathbf{k} - \mathbf{p}'$.

1. <u>The planar-box diagram:</u> We first rewrite the integral into the following form

$$\begin{aligned}
I_{\|} = &-(2\pi)^{-2} \int_{-\infty}^{+\infty} d\alpha \int_{-\infty}^{+\infty} d\beta \cdot \\
&\times \int_{-\infty}^{+\infty} dp_0' \int_{-\infty}^{+\infty} dp_0'' \int_{-\infty}^{+\infty} dp_0 \int_{-\infty}^{+\infty} dk_0 \int_{-\infty}^{+\infty} dk_0' \cdot \\
&\times \exp i\alpha(k_0' - p_0'' + p_0') \exp i\beta(p_0'' - p_0 + k_0) \cdot \\
&\times [p_0'^2 - A_{p'}^2 + i\delta]^{-1} [\omega^2 - k_0^2 - i\delta]^{-1} [\omega'^2 - k_0'^2 - i\delta]^{-1} \cdot \\
&\times [p_0''^2 - A_{p''}^2 + i\delta]^{-1} [p_0^2 - A_p^2 + i\delta]^{-1} \ .
\end{aligned} \tag{A2}$$

The energy-variable integrations can be performed in a straightforward manner, in principle, using the residue theorem, *e.g.*

$$\int_{-\infty}^{+\infty} dk_0 \ \frac{\exp i\beta k_0}{[\omega^2 - k_0^2 - i\delta]} = 2\pi i [2\omega]^{-1} e^{\mp i\beta\omega} \ , \tag{A3}$$

where in the exponential the (−)-sign and the (+)-sign apply to $\beta > 0$ respectively $\beta < 0$. Keeping track of the signs in the exponentials, the intermediate result is

$$\begin{aligned}
I_{\|} = &(2\pi)^{-2} (2\pi i)^5 \left[\ 32 \ \omega \ \omega' \ a \ a' \ a'' \ \right]^{-1} \cdot \\
&\times \left\{ \int_0^{\infty} d\alpha \int_0^{\alpha} d\beta e^{-i\beta(a+\omega)} e^{-i\alpha(a'+\omega')} e^{+i(\beta-\alpha)a''} \right. \\
&+ \int_0^{\infty} d\alpha \int_{\alpha}^{\infty} d\beta e^{-i\beta(a+\omega)} e^{-i\alpha(a'+\omega')} e^{-i(\beta-\alpha)a''} \\
&+ \int_{-\infty}^{0} d\alpha \int_0^{\infty} d\beta e^{-i\beta(a+\omega)} e^{+i\alpha(a'+\omega')} e^{-i(\beta-\alpha)a''} \\
&+ \int_{-\infty}^{0} d\alpha \int_{-\infty}^{\alpha} d\beta e^{+i\beta(a+\omega)} e^{+i\alpha(a'+\omega')} e^{+i(\beta-\alpha)a''} \\
&+ \int_{-\infty}^{0} d\alpha \int_{\alpha}^{0} d\beta e^{+i\beta(a+\omega)} e^{+i\alpha(a'+\omega')} e^{-i(\beta-\alpha)a''} \\
&+ \left. \int_0^{\infty} d\alpha \int_{-\infty}^{0} d\beta e^{+i\beta(a+\omega)} e^{-i\alpha(a'+\omega')} e^{+i(\beta-\alpha)a''} \right\} \ ,
\end{aligned} \tag{A4}$$

where for typografical reasons we have put $A_{p'} \equiv a'$, etc.
Performing now the remaining elementary integrals we find

$$I_{\parallel} = -(2\pi)^3 i \left[\frac{1}{4\omega\omega'} \frac{1}{4AA'}\right] \cdot$$
$$\times \Big\{ 2[A' + A'' + \omega']^{-1}[A + A' + \omega + \omega']^{-1}[A + A'' + \omega]^{-1}$$
$$+ \ 4[A' + A'' + \omega']^{-1}[2A'']^{-1}[A + A'' + \omega]^{-1} \Big\} \ . \tag{A5}$$

Here we have substituted for a-etc. again $A \equiv A_p$, etc.
The first term in the curly brackets corresponds diagrams (a) and (b) of Fig. (4), *i.e.* the planar BW-TPEP-graphs. The second term corresponds to diagrams (a-d) of Fig. (6), *i.e.* the TMO-graphs.
2. <u>The crossed-box diagram:</u> Here the integral can be written into the form

$$I_X = (2\pi)^{-3} \int_{-\infty}^{+\infty} d\alpha \int_{-\infty}^{+\infty} d\beta \int_{-\infty}^{+\infty} d\gamma \ \cdot$$
$$\times \int_{-\infty}^{+\infty} dp_0' \int_{-\infty}^{+\infty} dp_0'' \int_{-\infty}^{+\infty} dp_0''' \int_{-\infty}^{+\infty} dp_0 \int_{-\infty}^{+\infty} dk_0 \int_{-\infty}^{+\infty} dk_0' \ \cdot$$
$$\times \ e^{-i\alpha(k_0' - p_0 + p_0' + k_0)} e^{i\beta(p_0'' - p_0 + k_0)} e^{i\gamma(p_0''' + p_0' + k_0)} \ .$$
$$\times \ [p_0'^2 - A_{p'}^2 + i\delta]^{-1}[\omega^2 - k_0^2 - i\delta]^{-1}[\omega'^2 - k_0'^2 - i\delta]^{-1} \ \cdot$$
$$\times \ [p_0'' - A_{p''} + i\delta]^{-1}[p_0''' - A_{p'''} + i\delta]^{-1}[p_0^2 - A_p^2 + i\delta]^{-1} \ . \tag{A6}$$

Again the evaluation of the energy-variable integrals is done similarly as in the former case for the planar-box diagram. Here we get only contributions from $\beta < 0$ and $\gamma < 0$. We split the integral into two contributions

$$I_X = (2\pi)^{-3} \left(\int_{-\infty}^{0} d\beta \int_{-\infty}^{0} d\gamma \int_{0}^{+\infty} d\alpha + \int_{-\infty}^{0} d\beta \int_{-\infty}^{0} d\gamma \int_{-\infty}^{0} d\alpha \right) \times [\ldots] \ . \tag{A7}$$

For the first contribution we have $\alpha - \beta - \gamma > 0$, $\alpha - \beta > 0$, and $-\alpha + \gamma < 0$, which settles the signs in the exponentials. The residue theorem can be applied straightaway and one gets

$$I_X^{(1)} = (2\pi)^{-3}(2\pi i)^6 \Big[\ 16 \ \omega \ \omega' \ a \ a' \ \Big]^{-1} \int_{-\infty}^{0} d\beta \int_{-\infty}^{0} d\gamma \int_{0}^{\infty} d\alpha \ \cdot$$
$$\times \left\{ \ e^{+i\beta(a + a'' + \omega)} e^{+i\gamma(a' + a''' + \omega)} e^{-i\alpha(a' + \omega' + a + \omega)} \ \right\}$$
$$= -(2\pi)^3 i \left[\frac{1}{4\omega\omega'} \frac{1}{4AA'}\right] \cdot$$
$$\times \left\{ \ (A + A'' + \omega)^{-1}(A + A' + \omega + \omega')^{-1}(A''' + A' + \omega)^{-1} \ \right\} \ . \tag{A8}$$

In order to settle the signs in the exponentials for the second contribution, the region of integration has to be divided into eight separate regions. Doing the energy integrals we obtain the intermediate result

$$
\begin{aligned}
I_X^{(2)} &= -(2\pi)^3 \left[16\ \omega\ \omega'\ a\ a'\ i \right]^{-1} \cdot \\
&\times \left\{ \int_{-\infty}^{0} d\beta \int_{-\infty}^{\beta} d\gamma \int_{-\infty}^{\beta+\gamma} d\alpha\ e^{+i\alpha(a'+\omega'+a+\omega)} e^{+i\beta(a''-a-\omega)} e^{+i\gamma(a'''-a'-\omega)} \right. \\
&+ \int_{-\infty}^{0} d\beta \int_{-\infty}^{\beta} d\gamma \int_{\beta+\gamma}^{\gamma} d\alpha\ e^{+i\alpha(a'+a+\omega'-\omega)} e^{+i\beta(a''-a+\omega)} e^{+i\gamma(a'''-a'+\omega)} \\
&+ \int_{-\infty}^{0} d\beta \int_{-\infty}^{\beta} d\gamma \int_{\gamma}^{\beta} d\alpha\ e^{+i\alpha(-a'+a+\omega'-\omega)} e^{+i\beta(a''-a+\omega)} e^{+i\gamma(a'''+a'+\omega)} \\
&+ \int_{-\infty}^{0} d\beta \int_{-\infty}^{\beta} d\gamma \int_{\beta}^{0} d\alpha\ e^{+i\alpha(-a'-a+\omega'-\omega)} e^{+i\beta(a''+a+\omega)} e^{+i\gamma(a'''+a'+\omega)} \\
&+ \int_{-\infty}^{0} d\beta \int_{\beta}^{0} d\gamma \int_{-\infty}^{\beta+\gamma} d\alpha\ e^{+i\alpha(a'+a+\omega'+\omega)} e^{+i\beta(a''-a-\omega)} e^{+i\gamma(a'''-a'-\omega)} \\
&+ \int_{-\infty}^{0} d\beta \int_{\beta}^{0} d\gamma \int_{\beta+\gamma}^{\beta} d\alpha\ e^{+i\alpha(a'+a+\omega'-\omega)} e^{+i\beta(a''-a+\omega)} e^{+i\gamma(a'''-a'+\omega)} \\
&+ \int_{-\infty}^{0} d\beta \int_{\beta}^{0} d\gamma \int_{\beta}^{\gamma} d\alpha\ e^{+i\alpha(a'-a+\omega'-\omega)} e^{+i\beta(a''+a+\omega)} e^{+i\gamma(a'''-a'+\omega)} \\
&+ \left. \int_{-\infty}^{0} d\beta \int_{\beta}^{0} d\gamma \int_{\gamma}^{0} d\alpha\ e^{+i\alpha(-a'-a+\omega'-\omega)} e^{+i\beta(a''+a+\omega)} e^{+i\gamma(a'''+a'+\omega)} \right\} .
\end{aligned}
\tag{A9}
$$

The remaining integrals are again elementary and one gets

$$
\begin{aligned}
I_X^{(2)} &= -(2\pi)^3 i \left[\frac{1}{4\omega\omega'} \frac{1}{4AA'} \right] \cdot \\
&\cdot \left\{ (A'+A''+\omega')^{-1} (A+A'+\omega+\omega')^{-1} (A'''+A+\omega')^{-1} \right. \\
&+ (A'+A''+\omega')^{-1} (A''+A'''+\omega+\omega')^{-1} (A'''+A+\omega')^{-1} \\
&+ (A'''+A'+\omega)^{-1} (A''+A'''+\omega+\omega')^{-1} (A+A''+\omega)^{-1} \\
&+ (A'''+A'+\omega)^{-1} (A''+A'''+\omega+\omega')^{-1} (A+A'''+\omega')^{-1} \\
&+ \left. (A''+A'+\omega')^{-1} (A''+A'''+\omega+\omega')^{-1} (A+A''+\omega)^{-1} \right\} .
\end{aligned}
\tag{A10}
$$

The terms in $I_X^{(1)}$ and $I_X^{(2)}$ correspond to diagrams (a-f) of Fig. (5), *i.e.* the crossed BW-TPEP-graphs.

APPENDIX B

1. The Fourier transformation for OPE with a Gaussian form factor

$$I_2(m,r) \equiv (2\pi)^{-3}\int d^3k\ e^{i\mathbf{k}\cdot\mathbf{r}}\tilde{I}_2(\mathbf{k}^2)\ , \tag{B1}$$

with

$$\tilde{I}_2(\mathbf{k}^2) = \int_0^\infty d\mu^2 \frac{\rho(\mu^2)}{\mathbf{k}^2+\mu^2} \simeq \frac{e^{-\mathbf{k}^2/\Lambda^2}}{\mathbf{k}^2+m^2}\ , \tag{B2}$$

where we used the substitution of (6.4), has been given in [3] with the result

$$I_2(m,r) = \frac{m}{4\pi}\phi_C^0(m,r) \tag{B3}$$

$$\phi_C^0(m,r) = \exp(m^2/\Lambda^2)\frac{\left[e^{-mr}\ \mathrm{erfc}\ (-\frac{\Lambda r}{2}+\frac{m}{\Lambda}) - e^{mr}\mathrm{erfc}\ (\frac{\Lambda r}{2}+\frac{m}{\Lambda})\right]}{2mr}\ .$$

2. In order to deal with Fourier integrals where $\omega(\mathbf{k})^n, n=1,3,\ldots$ and/or powers of $\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)$ appear in the denominators, we exploit the following integral-representation

$$\frac{1}{\omega(\mathbf{k})} = \frac{2}{\pi}\int_0^\infty \frac{d\lambda}{\mathbf{k}^2+\mu^2+\lambda^2}\ , \tag{B4}$$

where $\omega(\mathbf{k}) = \sqrt{\mathbf{k}^2+\mu^2}$. Application for $1/\omega(\mathbf{k})$ gives

$$\begin{aligned}\tilde{I}_1(\mathbf{k}^2) &= \int_0^\infty d\mu^2 \frac{\rho(\mu^2)}{\sqrt{\mathbf{k}^2+\mu^2}} = \frac{2}{\pi}\int_0^\infty d\lambda \int_0^\infty d\mu^2 \frac{\rho(\mu^2)}{\mathbf{k}^2+\mu^2+\lambda^2} \\ &\simeq \frac{2}{\pi}\int_0^\infty d\lambda \frac{e^{-(\mathbf{k}^2+\lambda^2)/\Lambda^2}}{\mathbf{k}^2+\mu^2+\lambda^2}\ .\end{aligned} \tag{B5}$$

The Fourier transformation gives

$$I_1(m,r) \equiv (2\pi)^{-3}\int d^3k e^{i\mathbf{k}\cdot\mathbf{r}}\frac{e^{-\mathbf{k}^2/\Lambda^2}}{\omega(\mathbf{k})} = \frac{2}{\pi}\int_0^\infty d\lambda e^{-\lambda^2/\Lambda^2} I_2(\sqrt{m^2+\lambda^2},r)\ . \tag{B6}$$

Similarly, for the integral where $\omega^3(\mathbf{k})$ occurs in the denominator, using again the integral equation for $1/\omega(\mathbf{k})$ and the substitution of (6.4), we find

$$I_3(m,r) = \frac{2}{\pi}\int_0^\infty \frac{d\lambda}{\lambda^2}\left[I_2(m,r) - e^{-\lambda^2/\Lambda^2}I_2(\sqrt{m^2+\lambda^2},r)\right]\ . \tag{B7}$$

3. For integrals with $1/\omega^4(\mathbf{k})$ we find

$$\begin{aligned}\tilde{I}_4(\mathbf{k}^2) &= \int_0^\infty d\mu^2 \frac{\rho(\mu^2)}{(\mathbf{k}^2+\mu^2)^2} = -\frac{d}{d\mathbf{k}^2}\int_0^\infty d\mu^2 \frac{\rho(\mu^2)}{\mathbf{k}^2+\mu^2} \\ &\simeq \frac{e^{-\mathbf{k}^2/\Lambda^2}}{(\mathbf{k}^2+m^2)^2} + \frac{1}{\Lambda^2}\frac{e^{-\mathbf{k}^2/\Lambda^2}}{\mathbf{k}^2+m^2}\ .\end{aligned} \tag{B8}$$

The Fourier transformation gives

$$I_4(m,r) = -\frac{d}{dm^2}\ I_2(m,r) + \frac{1}{\Lambda^2}\ I_2(m,r)\ . \tag{B9}$$

4. Next we demonstrate that the separation of the variables $\mathbf{k}_1$ and $\mathbf{k}_2$ in the Fourier integrals that occur for Two-Pion-Exchange is always possible, allbeit at the cost of an integration over the variable λ. The method used by Lévy [24] can not be used in the case of Gaussian form factors. Fortunately, the integral representation given above for $1/\omega(\mathbf{k})$ enables us to achieve a factorisation of the integrands w.r.t. the $\mathbf{k}$-variables under the λ-parameter integral for all integrals that occur in the course of the evaluation of the TPE-potentials. This way we are left with only one-dimensional integrals for the TPE-potentials, which have to be evaluated numerically.

We consider the following typical integral

$$\tilde{J}_1(\mathbf{k}_1,\mathbf{k}_2) \;=\; \int_0^\infty d\mu_1^2 \int_0^\infty d\mu_2^2\ \frac{\rho(\mu_1^2)\rho(\mu_2^2)}{\omega(\mathbf{k}_1)\omega(\mathbf{k}_2)[\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2]}\ , \tag{B10}$$

which occurs for example in the course of the evaluation of the parallel box graph. Here $\omega(\mathbf{k}_1)=\sqrt{\mathbf{k}_1^2+\mu_1^2}$ and $\omega(\mathbf{k}_2)=\sqrt{\mathbf{k}_2^2+\mu_2^2}$. Now we employ the trick of Lévy [24] by writing

$$\frac{1}{\omega_1\omega_2}\frac{1}{\omega_1+\omega_2} = \left(\frac{1}{\omega_2}-\frac{1}{\omega_1}\right)\frac{1}{\omega_1^2-\omega_2^2}\ . \tag{B11}$$

Then, using the expression

$$\frac{1}{\omega_2}-\frac{1}{\omega_1} = \frac{2}{\pi}(\omega_1^2-\omega_2^2)\int_0^\infty \frac{d\lambda}{(\omega_1^2+\lambda^2)(\omega_2^2+\lambda^2)}\ , \tag{B12}$$

which can easily be derived, we obtain factorization of $\mathbf{k}_1$ and $\mathbf{k}_2$ under the λ-integral:

$$\tilde{J}_1(\mathbf{k}_1,\mathbf{k}_2) = \frac{2}{\pi}\int_0^\infty d\lambda \left[\int_0^\infty d\mu_1^2\ \frac{\rho(\mu_1^2)}{\mathbf{k}_1^2+\mu_1^2+\lambda^2}\right]\left[\int_0^\infty d\mu_2^2\ \frac{\rho(\mu_2^2)}{\mathbf{k}_2^2+\mu_2^2+\lambda^2}\right]\ . \tag{B13}$$

Using again the substitution in (6.4), we get

$$\tilde{J}_1(\mathbf{k}_1,\mathbf{k}_2) = \frac{2}{\pi}\int_0^\infty d\lambda \left[\frac{e^{-(\mathbf{k}_1^2+\lambda^2)/\Lambda^2}}{\mathbf{k}_1^2+m^2+\lambda^2}\right]\left[\frac{e^{-(\mathbf{k}_2^2+\lambda^2)/\Lambda^2}}{\mathbf{k}_2^2+m^2+\lambda^2}\right]\ . \tag{B14}$$

For the latter expression, the Fourier transformation

$$J_1(r) = (2\pi)^{-6}\int\int d^3k_1 d^3k_2\ e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}}\ \tilde{J}_1(\mathbf{k}_1,\mathbf{k}_2) \tag{B15}$$

can readily be performed. We finally arrive at the result

$$J_1(r) = \frac{2}{\pi}\int_0^\infty d\lambda e^{-2\lambda^2/\Lambda^2}\left[I_2(\sqrt{m^2+\lambda^2},r)\right]^2\ . \tag{B16}$$

Similarly we can treat all Fourier integrals that appear in the course of the calculation of the TPE-potentials. Notice that the tricks, employed here, also work in the case of the Two-Meson-Exchange Potentials, where in general the mesons have different masses.

In the point-coupling limit the J_1-function gives

$$\lim_{\Lambda\to\infty} J_1(r) = (m/r^2)K_1(mr)\,, \tag{B17}$$

which is the result of Lévy [24], Appendix B, Eq.(10a). Here K_1 is the modified Bessel function (see *e.g.* [40]).

In Table I we give the limit $\Lambda \to \infty$ for the functions given in this Appendix. They enable the reader to compare the potentials of this paper with those for point-couplings in the literature.

$\lim\limits_{\lambda\to\infty}$	F	F'	F''	F'''
$I_1(m,r)$	$\frac{m^2}{2\pi^2}\,K_1/x$	$-\frac{m^3}{2\pi^2}\left(K_0+\frac{2}{x}K_1\right)/x$	$\frac{m^4}{2\pi^2}\left(\frac{3}{x}K_0+(1+\frac{6}{x^2})K_1\right)/x$	
$I_2(m,r)$	$\frac{m}{4\pi}\frac{e^{-x}}{x}$	$-\frac{m^2}{4\pi}\left(1+\frac{1}{x}\right)\frac{e^{-x}}{x}$	$\frac{m^3}{4\pi}\left(1+\frac{2}{x}+\frac{2}{x^2}\right)\frac{e^{-x}}{x}$	$-\frac{m^4}{4\pi}\left(1+\frac{3}{x}+\frac{6}{x^2}+\frac{6}{x^3}\right)\frac{e^{-x}}{x}$
$I_3(m,r)$	$\frac{1}{2\pi^2}K_0$	$-\frac{m}{2\pi^2}K_1$	$\frac{m^2}{2\pi^2}\left(K_0+\frac{1}{x}K_1\right)$	$-\frac{m^3}{2\pi^2}\left(\frac{1}{x}K_0+(1+\frac{2}{x^2})K_1\right)$
$I_4(m,r)$	$\frac{1}{8\pi m}e^{-x}$	$-\frac{1}{8\pi}e^{-x}$	$\frac{m}{8\pi}e^{-x}$	$-\frac{m^2}{8\pi}e^{-x}$
$J_1(r)$	$m^3\,K_1/x$	$-m^4\left(\frac{1}{x}K_0+(1+\frac{2}{x^2})K_1\right)/x$	$m^5\left[(1+\frac{4}{x^2})K_0+(\frac{3}{x}+\frac{4}{x^2}+\frac{2}{x}d)K_1\right]/x$	

TABLE I. Limit $\Lambda \to \infty$ of the basic Fourier transforms. $K_0 \equiv K_0(x)$,$K_1 \equiv K_1(x)$, where $x = mr$, and $F' \equiv dF/dr$ etc.

APPENDIX C

In this Appendix we describe the relation between the instantaneous B-S wave-function and the Pauli-spinor Lippmann-Schwinger wave-functions.

The instantaneous B-S wave-function in momentum space is

$$\begin{aligned}\phi_{\alpha\beta}(\mathbf{p}) &= \int dp_0 \int d^4x e^{ipx}\left\langle 0\left|T\left[\psi_\alpha(\frac{x}{2})\psi_\beta(-\frac{x}{2})\right]\right|\Psi_{NN}\right\rangle \\ &= \int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}}\left\langle 0\left|\psi_\alpha(\frac{\mathbf{x}}{2})\psi_\beta\left(-\frac{\mathbf{x}}{2}\right)\right|\Psi_{NN}\right\rangle \end{aligned} \tag{C1}$$

where $|\Psi_{NN}\rangle$ denotes the two-nucleon scattering state. The expansion of the Schrödinger operators $\psi(\frac{\mathbf{x}}{2})$ and $\psi(-\frac{\mathbf{x}}{2})$ in annihilation and creation operators $b(\mathbf{p},s)$ and $d(\mathbf{p},s)$ reads

$$\begin{aligned}\psi(\frac{\mathbf{x}}{2}) = \sum_{s=\pm\frac{1}{2}}\int \frac{d^3q}{(2\pi)^{\frac{3}{2}}}\sqrt{\frac{m}{E(\mathbf{q})}}\cdot \\ \times\left[b(\mathbf{q},s)u(\mathbf{q},s)e^{i\mathbf{q}\cdot\mathbf{x}/2} + d^\dagger(\mathbf{q},s)v(\mathbf{q},s)e^{-i\mathbf{q}\cdot\mathbf{x}/2}\right]\cdot\end{aligned} \tag{C2}$$

Using this expansion in (C1) gives

$$\begin{aligned}
\phi_{++}(\mathbf{p}) &= \sum_{s_a,s_b} \langle 0|b(\mathbf{p},s_a;a)b(-\mathbf{p},s_b;b)|\Psi_{NN}\rangle u_a(\mathbf{p},s_a)\otimes u_b(-\mathbf{p},s_b) \ ,\\
\phi_{+-}(\mathbf{p}) &= \sum_{s_a,s_b} \langle 0|b(\mathbf{p},s_a;a)d^\dagger(-\mathbf{p},s_b;b)|\Psi_{NN}\rangle u_a(\mathbf{p},s_a)\otimes v_b(-\mathbf{p},s_b) \ ,\\
\phi_{-+}(\mathbf{p}) &= \sum_{s_a,s_b} \langle 0|d^\dagger(\mathbf{p},s_a;a)b(-\mathbf{p},s_b;b)|\Psi_{NN}\rangle v_a(\mathbf{p},s_a)\otimes u_b(-\mathbf{p},s_b) \ ,\\
\phi_{--}(\mathbf{p}) &= \sum_{s_a,s_b} \langle 0|d^\dagger(\mathbf{p},s_a;a)d^\dagger(-\mathbf{p},s_b;b)|\Psi_{NN}\rangle v_a(\mathbf{p},s_a)\otimes v_b(-\mathbf{p},s_b) \ .
\end{aligned} \tag{C3}$$

Introducing now the Lippmann-Schwinger wave-functions

$$\chi_{s_a s_b}(\mathbf{p}) \equiv \langle 0|b(\mathbf{p},s_a;a)b(-\mathbf{p},s_b;b)|\Psi_{NN}\rangle \ , \tag{C4}$$

we have

$$\phi_{++}(\mathbf{p}) = \sum_{s_a,s_b} \chi_{s_a s_b}(\mathbf{p}) \ u_a(\mathbf{p},s_a)\otimes u_b(-\mathbf{p},s_b) \ . \tag{C5}$$

Consider next Eq. (7.3) and write

$$\phi(\mathbf{p}) = \phi^{(0)}(\mathbf{p}) + \Lambda_+^{(a)}(\mathbf{p})\Lambda_+^{(b)}(-\mathbf{p}) \ \tilde{g}(\mathbf{p}) \int d^3p' \ V(\mathbf{p},\mathbf{p}') \ \phi(\mathbf{p}') \ . \tag{C6}$$

Using the identity

$$\Lambda_+(\mathbf{p}) = \sum_s u(\mathbf{p},s)\otimes \bar{u}(\mathbf{p},s) \tag{C7}$$

for the projection operators in (C6), and multiplying this equation on the left with $\bar{u}_a(\mathbf{p},s_a)\otimes\bar{u}_b(-\mathbf{p},s_b)$ we find, using the orthogonality relations for the Dirac spinors, the Lippmann-Schwinger equation for the Pauli-spinors

$$\chi_{s_a s_b}(\mathbf{p}) = \chi^{(0)}_{s_a s_b}(\mathbf{p}) + \tilde{g}(\mathbf{p}) \int d^3p' \ \mathcal{V}(\mathbf{p},\mathbf{p}')_{s_a s_b, s'_a s'_b} \ \chi_{s'_a s'_b}(\mathbf{p}') \ , \tag{C8}$$

where we defined

$$\begin{aligned}
\mathcal{V}(\mathbf{p},\mathbf{p}')_{s_a s_b, s'_a s'_b} &\equiv \chi^{(a)\dagger}_{s_a}\chi^{(b)\dagger}_{s_b} \ \mathcal{V} \ \chi^{(a)}_{s'_a}\chi^{(b)}_{s'_b}\\
&= \bar{u}_a(\mathbf{p},s_a)\bar{u}_b(-\mathbf{p},s_b) \ V(\mathbf{p},\mathbf{p}') \ u_a(\mathbf{p}',s'_a)u_b(-\mathbf{p}',s'_b) \ .
\end{aligned} \tag{C9}$$

Omitting now the spin labels in Eqs. (C8) and (C9), one arrives at the equations (7.6) of section VII.

APPENDIX D

To evaluate the momentum integrations, we write

$$e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{r}} = \lim_{\mathbf{r}_1\to\mathbf{r}_2} e^{i\mathbf{k}_1\cdot\mathbf{r}_1} e^{i\mathbf{k}_2\cdot\mathbf{r}_2} \tag{D1}$$

and take the limit operation before the momentum integrations. Then, we can replace all momenta, occurring in the numerator, by $\boldsymbol{\nabla}_1$- and $\boldsymbol{\nabla}_2$-operations, respectively the $\boldsymbol{\nabla}$-operations w.r.t. $\mathbf{r}_1$ and $\mathbf{r}_2$, and take these out of the momentum integrations.
For the evaluation of the potentials we employ the following useful identities

$$\text{(i)} \lim_{\mathbf{r}_1\to\mathbf{r}_2} (\boldsymbol{\nabla}_1\cdot\boldsymbol{\nabla}_2)^2 F(r_1)G(r_2) = \frac{2}{r^2}F'(r)G'(r) + F''(r)G''(r)\ , \tag{D2}$$

$$\begin{aligned}
&\text{(ii)} \lim_{\mathbf{r}_1\to\mathbf{r}_2} (\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla}_1\times\boldsymbol{\nabla}_2)(\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla}_1\times\boldsymbol{\nabla}_2)F(r_1)G(r_2) = \\
&\quad +\frac{2}{3}\left[\frac{1}{r^2}F'(r)G'(r) + \frac{1}{r}F'(r)G''(r) + \frac{1}{r}G'(r)F''(r)\right](\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2) \\
&\quad +\frac{1}{3}\left[\left(\frac{1}{r}F'(r) - F''(r)\right)\frac{1}{r}G'(r) + \frac{1}{r}F'(r)\left(\frac{1}{r}G'(r) - G''(r)\right)\right] S_{12}\ ,
\end{aligned} \tag{D3}$$

$$\begin{aligned}
&\text{(iii)} \lim_{\mathbf{r}_1\to\mathbf{r}_2} (\boldsymbol{\nabla}_1\cdot\boldsymbol{\nabla}_2)^3 F(r_1)G(r_2) = \\
&\quad \frac{6}{r^2}\left(\frac{1}{r}F'(r) - F''(r)\right)\left(\frac{1}{r}G'(r) - G''(r)\right) + F'''(r)G'''(r)\ ,
\end{aligned} \tag{D4}$$

$$\begin{aligned}
&\text{(iv)} \lim_{\mathbf{r}_1\to\mathbf{r}_2} (\boldsymbol{\nabla}_1\cdot\boldsymbol{\nabla}_2)(\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla}_1\times\boldsymbol{\nabla}_2)(\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla}_1\times\boldsymbol{\nabla}_2)F(r_1)G(r_2) = \\
&\quad -\frac{2}{3}\left[\frac{1}{r^2}\left(\frac{1}{r}F'(r) - F''(r)\right)\left(\frac{1}{r}G'(r) - G''(r)\right) + \left(\frac{1}{r}F'(r) - F''(r)\right)\frac{1}{r}G'''(r)\right. \\
&\quad \left.+\frac{1}{r}F'''(r)\left(\frac{1}{r}G'(r) - G''(r)\right)\right](\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2) \\
&\quad +\frac{1}{3}\left[\frac{4}{r^2}\left(\frac{1}{r}F'(r) - F''(r)\right)\left(\frac{1}{r}G'(r) - G''(r)\right) + \left(\frac{1}{r}F'(r) - F''(r)\right)\frac{1}{r}G'''(r)\right. \\
&\quad \left.+\frac{1}{r}F'''(r)\left(\frac{1}{r}G'(r) - G''(r)\right)\right] S_{12}\ .
\end{aligned} \tag{D5}$$

This list of formulas is not complete. However, all other cases are similar and can readily be worked out by the reader.

APPENDIX E

To derive the point-coupling limit of the TPEP's of this paper, we use the integral representations

$$K_0 = \int_0^\infty d\lambda\ \frac{e^{-\sqrt{m^2+\lambda^2}r}}{\sqrt{m^2+\lambda^2}} \quad , \quad K_1 = \frac{1}{m}\int_0^\infty d\lambda\ e^{-\sqrt{m^2+\lambda^2}r} \tag{E1}$$

and the relations

$$K_0'(x) = -K_1(x) \quad , \quad K_1'(x) = -K_0 - \frac{1}{x}K_1(x) \tag{E2}$$

where $x = mr$ and $K_0' = dK_0/dx$ etc.

Besides the limits given in Appendix B, we also need the following ones involving the functions F(r) and G(r), defined in (8.8)

$$\begin{aligned}
\lim_{\Lambda\to\infty}\int_0^\infty \frac{d\lambda}{\lambda^2}\,F(r) &= \frac{1}{4\pi}K_0(x) \ , \\
\lim_{\Lambda\to\infty}\int_0^\infty \frac{d\lambda}{\lambda^2}F'(r) &= -\frac{m}{4\pi}K_1(x) \ , \\
\lim_{\Lambda\to\infty}\int_0^\infty \frac{d\lambda}{\lambda^2}F''(r) &= \frac{m^2}{4\pi}\left[K_0(x) + \frac{1}{x}K_1(x)\right] \ , \\
\lim_{\Lambda\to\infty}\int_0^\infty \frac{d\lambda}{\lambda^2}F(r)G(r) &= \frac{m}{(4\pi)^2}\left[\frac{2}{x}K_0(2x) - \frac{e^{-x}}{x}K_0(x)\right] \ , \\
\lim_{\Lambda\to\infty}\int_0^\infty \frac{d\lambda}{\lambda^2}F'(r)G'(r) &= -\frac{m^3}{(4\pi)^2}\left[\left(1+\frac{1}{x}\right)\frac{e^{-x}}{x}K_1(x) - \frac{2}{x}K_0(2x) - \frac{3}{x^2}K_1(2x)\right] \ , \\
\lim_{\Lambda\to\infty}\int_0^\infty \frac{d\lambda}{\lambda^2}F''(r)G'(r) &= \frac{m^4}{(4\pi)^2}\left[\left(1+\frac{2}{x}+\frac{2}{x^2}\right)\frac{e^{-x}}{x}K_1(x) - \frac{4}{x^2}K_0(2x) + \right. \\
&\qquad \left. -\left(2+\frac{9}{2x^2}\right)\frac{1}{x}K_1(2x)\right] \ , \\
\lim_{\Lambda\to\infty}\int_0^\infty \frac{d\lambda}{\lambda^2}F'(r)G''(r) &= \frac{m^4}{(4\pi)^2}\left[-\left(1+\frac{1}{x}\right)\frac{e^{-x}}{x}K_1'(x) - \frac{4}{x^2}K_0(2x) + \right. \\
&\qquad \left. -\left(2+\frac{9}{2x^2}\right)\frac{1}{x}K_1(2x)\right] \ , \\
\lim_{\Lambda\to\infty}\int_0^\infty \frac{d\lambda}{\lambda^2}F''(r)G''(r) &= \frac{m^5}{(4\pi)^2}\left[\left(1+\frac{2}{x}+\frac{2}{x^2}\right)\frac{e^{-x}}{x}K_1'(x) + \left(\frac{2}{x}+\frac{15}{2x^3}\right)K_0(2x) + \right. \\
&\qquad \left. +\left(\frac{6}{x^2}+\frac{11}{2x^4}\right)K_1(2x)\right] \ .
\end{aligned} \tag{E3}$$

where $F' \equiv dF/dr$ etc. and $K_1' = dK_1/dx$.

With these results and Table I one finds for the potentials in (8.7) the point-coupling limits

$$\begin{aligned}
\bar{V}_C^{(1)}(BW) = -m\left(\frac{f^2}{4\pi}\right)^2 \frac{2}{\pi}\Bigg\{ & 3\left[\left(1+\frac{2}{x}+\frac{2}{x^2}\right)K_0(x) + \left(1+\frac{4}{x}+\frac{4}{x^2}\right)\frac{1}{x}K_1(x)\right]\frac{e^{-x}}{x} + \\
& +\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\left[-2\left\{\left(1+\frac{2}{x}+\frac{2}{x^2}\right)K_0(x) + \left(1+\frac{4}{x}+\frac{4}{x^2}\right)\frac{1}{x}K_1(x)\right\}\frac{e^{-x}}{x} + \right.
\end{aligned}$$

$$+\frac{1}{x}\left\{\left(4+\frac{23}{x^2}\right)K_0(2x)+\left(12+\frac{23}{x^2}\right)\frac{1}{x}K_1(2x)\right\}\Bigg]\ \Bigg\}\ ,$$

$$\bar{V}_\sigma^{(1)}(BW) = -m\left(\frac{f^2}{4\pi}\right)^2\frac{2}{\pi}\Bigg\{\ 2\Bigg[\left\{\left(\frac{1}{x}+\frac{1}{x^2}\right)K_0(x)+\left(1+\frac{2}{x}+\frac{2}{x^2}\right)\frac{1}{x}K_1(x)\right\}\frac{e^{-x}}{x}+$$
$$-\frac{6}{x^3}K_0(2x)-\left(\frac{4}{x}+\frac{6}{x^3}\right)\frac{1}{x}K_1(2x)\Bigg]-\frac{4}{3}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\left[\left(\frac{1}{x}+\frac{1}{x^2}\right)K_0(x)+\right.$$
$$\left.+\left(1+\frac{2}{x}+\frac{2}{x^2}\right)\frac{1}{x}K_1(x)\right]\frac{e^{-x}}{x}\ \Bigg\}\ ,$$

$$\bar{V}_T^{(1)}(BW) = -m\left(\frac{f^2}{4\pi}\right)^2\frac{2}{\pi}\Bigg\{\ \Bigg[-\left\{\left(\frac{1}{x}+\frac{1}{x^2}\right)K_0(x)+\left(1+\frac{5}{x}+\frac{5}{x^2}\right)\frac{1}{x}K_1(x)\right\}\frac{e^{-x}}{x}+$$
$$+\frac{12}{x^3}K_0(2x)+\left(\frac{4}{x}+\frac{15}{x^3}\right)\frac{1}{x}K_1(2x)\Bigg]+\frac{2}{3}\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2\left[\left(\frac{1}{x}+\frac{1}{x^2}\right)K_0(x)+\right.$$
$$\left.+\left(1+\frac{5}{x}+\frac{5}{x^2}\right)\frac{1}{x}K_1(x)\right]\frac{e^{-x}}{x}\ \Bigg\}\ ,\tag{E4}$$

where we have introduced the notation $\bar{V}\equiv\lim_{\Lambda\to\infty}V$. These potentials are the same as the BW-potential v_4 in Eq. (61) of [25].

For the $1/M$-potentials, the use of Table I leads to the following contributions in the point-coupling limit:

$$\bar{V}_C^{(2)}(BW) = \quad(3+2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{m^2}{M}\right)\left(\frac{f^2}{4\pi}\right)^2\left(1+\frac{1}{x}\right)^2\frac{e^{-2x}}{x^2}\ ,$$
$$\bar{V}_{SO}^{(2)}(BW) = -(3+2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{2m^2}{M}\right)\left(\frac{f^2}{4\pi}\right)^2\left(1+\frac{1}{x}\right)\left(1+\frac{1}{x}+\frac{1}{x^2}\right)\frac{e^{-2x}}{x^3}\ .\tag{E5}$$

These potentials are the same as those of [27], paper II, Eq. (3).

$$\bar{V}_C^{(3)}(BW) = \quad(3+2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{m^2}{2M}\right)\left(\frac{f^2}{4\pi}\right)^2\left(1+\frac{3}{x}+\frac{12}{x^2}+\frac{30}{x^3}+\frac{36}{x^4}+\frac{18}{x^5}\right)\frac{e^{-2x}}{x}\ ,$$
$$\bar{V}_\sigma^{(3)}(BW) = -(3+2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{m^2}{2M}\right)\frac{2}{3}\left(\frac{f^2}{4\pi}\right)^2\left(2+\frac{8}{x}+\frac{16}{x^2}+\frac{18}{x^3}+\frac{9}{x^4}\right)\frac{e^{-2x}}{x^2}\ ,$$
$$\bar{V}_T^{(3)}(BW) = \quad(3+2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{m^2}{2M}\right)\frac{1}{3}\left(\frac{f^2}{4\pi}\right)^2\left(2+\frac{11}{x}+\frac{28}{x^2}+\frac{36}{x^3}+\frac{18}{x^4}\right)\frac{e^{-2x}}{x^2}\ .\tag{E6}$$

These potentials correspond to the non-adiabatic terms from the crossed-box diagrams as given in [27], paper I, Eq. (22). To compare, one has to subtract from (E6), Eq. (21) of [27], which is equal to the expressions in (E8) below.

For the TMO-potentials of section IX, the expressions of Table I lead to the following contributions in the point-coupling limit:

$$\begin{aligned}
\bar{V}_C^{(1)}(TMO) = &\quad m\,(3-2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{f^2}{4\pi}\right)^2 \frac{2}{\pi}\left[\left(1+\frac{2}{x}+\frac{2}{x^2}\right)K_0(x)+\right.\\
&\left.+\left(1+\frac{4}{x}+\frac{4}{x^2}\right)\frac{1}{x}K_1(x)\right]\frac{e^{-x}}{x}\ ,\\
\bar{V}_\sigma^{(1)}(TMO) = &\quad \frac{2m}{3}\,(3-2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{f^2}{4\pi}\right)^2 \frac{2}{\pi}\left[\left(\frac{1}{x}+\frac{1}{x^2}\right)K_0(x)+\right.\\
&\left.+\left(1+\frac{2}{x}+\frac{2}{x^2}\right)\frac{1}{x}K_1(x)\right]\frac{e^{-x}}{x}\ ,\\
\bar{V}_T^{(1)}(TMO) = &-\frac{m}{3}\,(3-2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{f^2}{4\pi}\right)^2 \frac{2}{\pi}\left[\left(\frac{1}{x}+\frac{1}{x^2}\right)K_0(x)+\right.\\
&\left.+\left(1+\frac{5}{x}+\frac{5}{x^2}\right)\frac{1}{x}K_1(x)\right]\frac{e^{-x}}{x}\ .
\end{aligned}\tag{E7}$$

The $\bar{V}^{(1)}(TMO)$-potentials are identical to those as given in, for example [1], p. 114.

$$\begin{aligned}
\bar{V}_C^{(2)}(TMO) &= -\,(3-2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{f^2}{4\pi}\right)^2 \frac{m^2}{4M}\left(1+\frac{3}{x}+\frac{12}{x^2}+\frac{30}{x^3}+\frac{36}{x^4}+\frac{18}{x^5}\right)\frac{e^{-2x}}{x}\ ,\\
\bar{V}_\sigma^{(2)}(TMO) &= -\,(3-2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{f^2}{4\pi}\right)^2 \frac{m^2}{3M}\left(1+\frac{4}{x}+\frac{8}{x^2}+\frac{9}{x^3}+\frac{9}{2x^4}\right)\frac{e^{-2x}}{x^2}\ ,\\
\bar{V}_T^{(2)}(TMO) &= \quad (3-2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\left(\frac{f^2}{4\pi}\right)^2 \frac{m^2}{6M}\left(1+\frac{11}{2x}+\frac{14}{x^2}+\frac{18}{x^3}+\frac{9}{x^4}\right)\frac{e^{-2x}}{x^2}\ .
\end{aligned}\tag{E8}$$

The $\bar{V}^{(2)}(TMO)$-potentials correspond to the potential of [27], paper I, Eq. (21).

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FIG. 1. Feynman-diagram One-Pion-Exchange.

FIG. 2. Feynman-diagrams Two-Pion-Exchange.

FIG. 3. Second-order Potential graphs.

FIG. 4. Planar BW-Two-Pion-Exchange Potential graphs. The arrows correspond to the definition of $\mathbf{k}_1$ and $\mathbf{k}_2$. This same convention is also used in all other graphs.

FIG. 5. Crossed BW-Two-Pion-Exchange Potential graphs.

FIG. 6. TMO-Two-Pion-Exchange Potential graphs.

FIG. 7. Second- and Fourth-order Potential Scattering Diagrams.

FIG. 8. Central $I = 0$ and $I = 1$ TPEP with the BW- and TMO-graph contributions.

FIG. 9. Spin-spin $I = 0$ and $I = 1$ TPEP with the BW- and TMO-graph contributions.

FIG. 10. Tensor $I = 0$ and $I = 1$ TPEP with the BW- and TMO-graph contributions.

FIG. 11. Spin-orbit $I = 0$ and $I = 1$ TPEP with the BW- and TMO-graph contributions.

FIG. 12. Central $I = 0$ and $I = 1$ TPEP-tail versus OPEP and HBEP.

FIG. 13. Spin-spin $I = 0$ and $I = 1$ TPEP-tail versus OPEP and HBEP.

FIG. 14. Tensor $I = 0$ and $I = 1$ TPEP-tail versus OPEP and HBEP.

FIG. 15. Spin-orbit $I = 0$ and $I = 1$ TPEP-tail versus OPEP and HBEP.