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Soft-Core OBE-Potentials in Momentum Space *

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Abstract

The partial wave projection of the Nijmegen soft-core potentials in momentum space is presented. The given formulas are quite general and apply to nucleon-nucleon and hyperon-nucleon as well. An important aspect of this work is that the results of the Nijmegen multi-energy partial wave analysis can readily be made available to momentum space computations. This by a straightforward transcription to momentum space of the various Nijmegen potentials based on soft-core potential functions. Typical structures of the momentum space potentials are shown grafically in three-dimensional plots for various nucleon-nucleon partial waves.

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I. INTRODUCTION

In the late fifties, the combination of nucleon-nucleon (NN) partial wave analyses (PWA) and endeavors to construct theoretical models led to the establisment of a largely phenomenological spin-orbit NN-force. For a clear account of this development see the papers by Signell and Marshak [1] and the references quoted therein. A modern and sophisticated version of the methodical exploitation of semi-phenomenological potentials in the PWA of the NN-data is provided by the work of the Nijmegen group. After the multi-energy PWA (*i.e.* phase shift analysis) of the pp data below 350 MeV [2], recently a similar analysis has been published for the $pp+np$ data [3]. The $pp+np$ data base consists of 1787 pp-data and 2514 np-data. Here, the internucleonic distance r_{NN} is divided up into three regions: (i) the long range region ($r_{NN} \geq 2.0$ fm), (ii) the intermediate region ($b \leq r_{NN} \leq 2.0$ fm, $b = 1.4$ fm), and (iii) the short range region ($r_{NN} \leq b$ fm). In (i) the NN-potential is very much dominated by the known electromagnetic and one-pion-exchange potentials, in (ii) are added intermediate range potentials that are parameterized by (broad) scalar and vector meson exchanges. In [3] the Nijmegen soft-core potential [4] was used, apart from a modification in the singlet waves. The short range region is described by energy dependent square well potentials. The well-depths are parametrized as polynomials in k^2, where k denotes the relativistic cm-momentum, independently for each partial wave. As a result of this PWA very accurate $I = 0$ np phases are now available, the estimated errors are only slightly larger than those for the pp phases.

The Nijmegen group has also performed PWA's using a Reidlike approach [5] to the construction of a phenomenological potential [6]. These Reidlike potentials fit the data equally well as the PWA's of [2,3], giving $\chi^2_{\rm p.d.p} \approx 1.0$. The function of these phenomenological potentials is to make the results of this new PWA available for many applications in few body systems. In these Reidlike potentials, an important place is taken by the so-called soft-core one-boson-exchange (OBE) potentials. For nucleon-nucleon [4] and hyperon-nucleon (YN) [7] scattering we have shown that a soft-core One-Boson-Exchange (OBE) model, based on Regge-pole theory, gives a good description of the NN and YN data. Recently the Nijmegen NN-potential, Nijm78, of [4] was updated [6] to Nijm93. These OBE-models were evaluated in configuration space through a fit to the data. In order to make the soft-core models also available in momentum space, we present in this contribution the explicit formulas for that purpose on the LSJ-partial wave basis.

The contents of this paper are as follows. In section II we review the definition of the OBE-potentials in the context of the Lippmann-Schwinger equation. We introduce the usual potential forms in Pauli spinor space, where we include the central (C), the spin-spin (σ), the tensor (T), the spin-orbit (SO), the quadratic spin-orbit (Q_{12}), and the antisymmetric spin-orbit (ASO) potentials. To make this paper self-contained we give in section III the OBE-potentials in momentum space for pseudo-scalar, vector, scalar, and diffractive exchanges. In section IV we perform the basic partial wave projections, in particular those for the spinor invariants. The partial wave basis is chosen according to the convention of [8]. In section V the exact relation between the configuration space and momentum space potentials in the case of the quadratic spin-orbit operator Q_{12} is discussed for the Nijmegen soft-core models [4,7]. Here the explicit corrections due to the difference between the $P_5 = (\boldsymbol{\sigma}_1 \cdot \mathbf{n})(\boldsymbol{\sigma}_2 \cdot \mathbf{n})$-operator and the Fourier transform of Q_{12} are given. Finally in section VI we discuss the tests performed on the formulas of this paper and give the results for the nucleon-nucleon s- and p-waves.

Appendix A contains sets of expansion coefficients in $z = \cos\theta$ for the potentials of the different exchanges. Appendix B contains partial wave matrix elements of several important operators. In appendix C the details are given of the Fourier transformation of the Q_{12}-potentials. In appendix D coefficients are given for the partial wave projection of the quadratic spin-orbit operator. The latter are introduced to make the expressions for the potentials less cumbersome. In appendices E–H we give the explicit momentum space partial wave potentials for respectively the pseudo-scalar-, the vector-, the scalar-, and the so-called diffractive-exchanges. In the latter we include the pomeron- (or multi-gluon-) exchange as well as the $J = 0$-components of the tensor-meson exchange.

II. POTENTIALS FOR THE LIPPMANN-SCHWINGER EQUATION

We consider the nucleon-nucleon or hyperon-nucleon reactions

$$B(p_1, s_1) + N(p_2, s_2) \to B'(p_1', s_1') + N'(p_2', s_2') \ . \tag{1}$$

where B is either N or Y. Like in [9], whose conventions we will follow in this paper, we will refer to B and B' as particles 1 and 3 and to N and N' as particles 2 and 4. The four momentum of particle i is $p_i = (E_i, \mathbf{p}_i)$ where $E_i = \sqrt{\mathbf{p}_i^2 + M_i^2}$ and M_i is the mass. The transition amplitude matrix M is related to the S-matrix via

$$\langle f|S|i\rangle = \langle f|i\rangle - i(2\pi)^4\delta^4(P_f - P_i)\langle f|M|i\rangle \ , \tag{2}$$

where $P_i = p_1 + p_2$ and $P_f = p_1' + p_2'$ represent the total four momentum for the initial state $|i\rangle$ and the final state $|f\rangle$. The latter refer to the two-particle states, which we normalize in the following way

$$\langle \mathbf{p}_1', \mathbf{p}_2'|\mathbf{p}_1, \mathbf{p}_2\rangle = (2\pi)^3 2E(\mathbf{p}_1)\delta^3(\mathbf{p}_1' - \mathbf{p}_1) \cdot (2\pi)^3 2E(\mathbf{p}_2)\delta^3(\mathbf{p}_2' - \mathbf{p}_2) \ . \tag{3}$$

Three-dimensional integral equations for the amplitudes $\langle f|M|i\rangle$ can be derived in various ways. See for example references [9,10]. In [11] the derivation is based entirely on two-particle unitarity and the analyticity properties of the amplitudes, using the N/D-formalism. In the latter approach the in essence Regge pole nature of meson-exchange can be apprehended most easily. The equation obtained with this method is

$$\begin{aligned} M_{fi}(\mathbf{q}_f, \mathbf{q}_i; s) &= W_{fi}(\mathbf{q}_f, \mathbf{q}_i; s) + \\ &+ \sum_n \int \frac{d^3k_n}{(2\pi)^3} W_{fn}(\mathbf{q}_f, \mathbf{k}_n; s) G_0(\mathbf{k}_n, s) M_{ni}(\mathbf{k}_n, \mathbf{q}_i; s) \ , \end{aligned} \tag{4}$$

where $\mathbf{q}_i$ and $\mathbf{q}_f$ denote the initial and final state momenta, and

$$G_0(\mathbf{k}; s) = \frac{1}{2}\frac{E_1(\mathbf{k}) + E_2(\mathbf{k})}{E_1(\mathbf{k})E_2(\mathbf{k})}\left[s - (E_1(\mathbf{k}) + E_2(\mathbf{k}))^2 + i\varepsilon\right]^{-1} \ , \tag{5}$$

with $s = (E_1(\mathbf{p}) + E_2(\mathbf{p}))^2$. This follows from equation (4.27) in [11]. The same equation has been derived, for example, by Gersten, Verhoeven, and de Swart [12] in the context of the conventional approach which uses the Bethe-Salpeter equation. Also in [11] it is shown in detail that in the Regge pole approximation the pseudopotential $\langle f|W|i\rangle$ corresponds to OBE-exchange amplitudes with form factors at the BBM-vertices. Beyond this, one may consider the OBE-approximation more generally as an effective way to represent the exchange amplitudes for all allowed quantum numbers. In order to arrive at a Lippmann-Schwinger equation, one chooses a new Green-function $g(\mathbf{k};s)$ which satisfies a dispersion relation in $\mathbf{q}^2(s)$ rather than in s [9]. Then one obtains

$$g(\mathbf{k}_n;s) = \frac{-1}{2[E_1(\mathbf{k}_n) + E_2(\mathbf{k}_n)]}(\mathbf{k}_n^2 - \mathbf{q}_n^2 - i\varepsilon)^{-1} , \tag{6}$$

where $\mathbf{q}_n$ is the on-energy-shell momentum. This Green-function is eventually used in the integral equation (4) instead of $G_0(\mathbf{k}_n;s)$. So the corrections to $\langle f|W|i\rangle$ due to the transformation of the Green-functions are neglected. They are of higher order in the couplings and are usually discarded in an OBE-approach. With the substitution of g for G, (5) becomes identical to equation (2.19) of [9]. From now on we follow section II of [9] in detail. The transformation to the non-relativistic normalization of the two-particle states leads to states with

$$(\mathbf{p}_1', s_1'; \mathbf{p}_2', s_2'|\mathbf{p}_1, s_1; \mathbf{p}_2, s_2) = (2\pi)^6\delta^3(\mathbf{p}_1' - \mathbf{p}_1)\delta^3(\mathbf{p}_2' - \mathbf{p}_2)\delta_{s_1',s_1}\delta_{s_2',s_1} . \tag{7}$$

For these states we define the T-matrix by

$$(f|T|i) = \{4\mu_{34}(E_3 + E_4)\}^{-\frac{1}{2}}\langle f|M|i\rangle\{4\mu_{12}(E_1 + E_2)\}^{-\frac{1}{2}} , \tag{8}$$

where μ_{12} and μ_{34} are the reduced masses for respectively the initial and final state. Then we get from (4) the Lippmann-Schwinger equation

$$\begin{aligned}(3,4|T|1,2) &= (3,4|V|1,2) + \\ &+ \sum_n \int \frac{d^3k_n}{(2\pi)^3}(3,4|V|n_1,n_2)\frac{2\mu_{n_1,n_2}}{\mathbf{q}_n^2 - \mathbf{k}_n^2 + i\varepsilon}(n_1,n_2|T|1,2) ,\end{aligned} \tag{9}$$

where analogously to (8), the potential V is defined as

$$(f|V|i) = \{4\mu_{34}(E_3 + E_4)\}^{-\frac{1}{2}}\langle f|W|i\rangle\{4\mu_{12}(E_1 + E_2)\}^{-\frac{1}{2}} . \tag{10}$$

Using rotational invariance and parity conservation we expand the T-matrix, which is a 4×4-matrix in Pauli-spinor space, into a complete set of Pauli-spinor invariants (see for example [7,13,14])

$$T = \sum_{\alpha=1}^{8} T_\alpha(\mathbf{q}_f^2, \mathbf{q}_i^2, \mathbf{q}_i\cdot\mathbf{q}_f)\ P_\alpha . \tag{11}$$

Introducing

$$\mathbf{q} = \frac{1}{2}(\mathbf{q}_f + \mathbf{q}_i) \ , \qquad \mathbf{k} = \mathbf{q}_f - \mathbf{q}_i \ , \qquad \mathbf{n} = \mathbf{q}_i \times \mathbf{q}_f = \mathbf{q} \times \mathbf{k} \ , \tag{12}$$

we choose for the operators P_α in spin-space

$$\begin{array}{ll} P_1 = 1 & P_2 = \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \\ P_3 = (\boldsymbol{\sigma}_1 \cdot \mathbf{k})(\boldsymbol{\sigma}_2 \cdot \mathbf{k}) - \frac{1}{3}(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\mathbf{k}^2 & P_4 = \frac{i}{2}(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \cdot \mathbf{n} \\ P_5 = (\boldsymbol{\sigma}_1 \cdot \mathbf{n})(\boldsymbol{\sigma}_2 \cdot \mathbf{n}) & P_6 = \frac{i}{2}(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) \cdot \mathbf{n} \\ P_7 = (\boldsymbol{\sigma}_1 \cdot \mathbf{q})(\boldsymbol{\sigma}_2 \cdot \mathbf{k}) + (\boldsymbol{\sigma}_1 \cdot \mathbf{k})(\boldsymbol{\sigma}_2 \cdot \mathbf{q}) & \\ P_8 = (\boldsymbol{\sigma}_1 \cdot \mathbf{q})(\boldsymbol{\sigma}_2 \cdot \mathbf{k}) - (\boldsymbol{\sigma}_1 \cdot \mathbf{k})(\boldsymbol{\sigma}_2 \cdot \mathbf{q}) & \end{array} \tag{13}$$

Here we follow [7], where in contrast to [4], we have chosen P_3 to be a purely 'tensor-force' operator.

In the OBEP-approximation only second-order irreducible diagrams contributing to the kernel *i.e.* $W = M^{\mathrm{irr}(2)}$ are included. Similarly to (11) we expand the potentials V. Again following [7], we neglect the potential forms P_7 and P_8, and also the dependence of the potentials on $\mathbf{k} \cdot \mathbf{q}$. Then, the expansion (11) reads for the potentials as follows

$$V = \sum_{\alpha=1}^{6} V_\alpha(\mathbf{k}^2, \mathbf{q}^2) \ P_\alpha \ . \tag{14}$$

III. ONE-BOSON-EXCHANGE POTENTIALS IN MOMENTUM SPACE

For completeness we will present the NN- and YN-potentials as derived in [4] and [7]. The local interaction Hamilton densities for the different couplings are
a) Pseudoscalar-meson exchange

$$\mathcal{H}_{PV} = \frac{f_P}{m_S} \ [i\bar{\psi}\gamma_\mu\gamma_5\psi]\partial^\mu\phi_P \ , \tag{15}$$

b) Vector-meson exchange

$$\mathcal{H}_V = g_V[i\bar{\psi}\gamma_\mu\psi]\phi_V^\mu + \frac{f_V}{4\mathcal{M}} \ [\bar{\psi}\sigma_{\mu\nu}\psi](\partial^\mu\phi_V^\nu - \partial^\nu\phi_V^\mu) \ , \tag{16}$$

c) Scalar-meson exchange

$$\mathcal{H}_S = g_S[\bar{\psi}\psi]\phi_S \ , \tag{17}$$

where $\sigma_{\mu\nu} = [\gamma_\mu, \gamma_\nu]/2i$ and m_S and $\mathcal{M}$ are scaling masses. In [4] and [7] the latter were chosen to be the charged pion mass and the proton mass, respectively. The vertices for 'diffractive'-exchange have the same Lorentz structure as those for scalar-meson-exchange. Including form factors $f(\mathbf{x}' - \mathbf{x})$, the interaction densities are modified to

$$H_X(\mathbf{x}) = \int d^3x' \, f(\mathbf{x}' - \mathbf{x})\mathcal{H}_X(\mathbf{x}') \ , \tag{18}$$

where $X = PV,\ V,\ S$, or D. Because of this 'convolutive' form, the potentials in momentum space are the same as for point interactions, except that the coupling constants are multiplied by the Fourier transform of the form factors.

The OBE-potentials were obtained in the standard way (see *e.g.* [4] and [7]) by evaluating the NN-interaction in Born-approximation. We write the potentials V_α of (14) in the form

$$V_\alpha(\mathbf{k}^2, \mathbf{q}^2) = \sum_X \Omega_\alpha^{(X)}(\mathbf{k}^2, \mathbf{q}^2) \cdot \Delta^{(X)}(\mathbf{k}^2, m^2, \Lambda^2)\ , \tag{19}$$

where $X = P,\ V,\ S$, and D (P = pseudo-scalar, V = vector, S = scalar, and D = diffractive). Furthermore

$$\Delta^{(X)}(\mathbf{k}^2, m^2, \Lambda^2) = \frac{1}{\mathbf{k}^2 + m^2} \cdot e^{-\mathbf{k}^2/\Lambda^2} \tag{20}$$

for $X = P,\ V,\ S$, and

$$\Delta^{(X)}(\mathbf{k}^2, m_P^2, \Lambda^2) = \frac{1}{\mathcal{M}^2} e^{-\mathbf{k}^2/(4m_P^2)} \tag{21}$$

for $X = D$. In (21) $\mathcal{M}$ is a universal scaling mass, which is in principle different from the one introduced in (16). In the YN-model [7] $\mathcal{M}$ was taken to be the proton mass [15]. The mass parameter m_P controls the $\mathbf{k}^2$-dependence of the pomeron-, and the $J = 0$-components of the f-, f'-, and A_2-potentials.

In [4] and [7] the following contributions to the different $\Omega_\alpha^{(X)}$ were derived:
a) pseudo-scalar-meson exchange:

$$\begin{aligned}
\Omega_2^{(P)} &= -f_{13}^P f_{24}^P \frac{\mathbf{k}^2}{3m_S^2} = -g_{13}^P g_{24}^P \left(\frac{\mathbf{k}^2}{12M_{13}M_{24}}\right) \\
\Omega_3^{(P)} &= -f_{13}^P f_{24}^P \frac{1}{m_S^2} = -g_{13}^P g_{24}^P \left(\frac{1}{4M_{13}M_{24}}\right)
\end{aligned} \tag{22}$$

We have also included here the expressions for the PS-coupling for completeness.
b) vector-meson exchange:

$$\begin{aligned}
\Omega_1^{(V)} &= \left\{ g_{13}^V g_{24}^V \left(1 - \frac{\mathbf{k}^2}{8M_{13}M_{24}} + \frac{3\mathbf{q}^2}{2M_{13}M_{24}}\right) \right. \\
&\qquad \left. - g_{13}^V g_{24}^V \frac{\mathbf{k}^2}{4\mathcal{M}M_{24}} - f_{13}^V g_{24}^V \frac{\mathbf{k}^2}{4\mathcal{M}M_{13}} + f_{13}^V f_{24}^V \frac{\mathbf{k}^4}{16\mathcal{M}^2 M_{13}M_{24}} \right\} \\
\Omega_2^{(V)} &= -\frac{2}{3}\,\mathbf{k}^2\,\Omega_3^{(V)} \\
\Omega_3^{(V)} &= \left\{ \left(g_{13}^V + f_{13}^V \frac{M_{13}}{\mathcal{M}}\right)\left(g_{24}^V + f_{24}^V \frac{M_{24}}{\mathcal{M}}\right) - f_{13}^V f_{24}^V \frac{\mathbf{k}^2}{8\mathcal{M}^2} \right\} / (4M_{13}M_{24}) \\
\Omega_4^{(V)} &= -\left\{ 12 g_{13}^V g_{24}^V + 8(g_{13}^V f_{24}^V + f_{13}^V g_{24}^V) \frac{\sqrt{M_{13}M_{24}}}{\mathcal{M}} \right.
\end{aligned}$$

$$-f^V_{13}f^V_{24}\frac{3\mathbf{k}^2}{\mathcal{M}^2}\Bigg\}/(8M_{13}M_{24})$$

$$\Omega^{(V)}_5 = -\Bigg\{g^V_{13}g^V_{24} + 4(g^V_{13}f^V_{24} + f^V_{13}g^V_{24})\frac{\sqrt{M_{13}M_{24}}}{\mathcal{M}}$$

$$+8f^V_{13}f^V_{24}\frac{M_{13}M_{24}}{\mathcal{M}^2}\Bigg\}/(16M^2_{13}M^2_{24})$$

$$\Omega^{(V)}_6 = -\Bigg\{\left(g^V_{13}g^V_{24} + f^V_{13}f^V_{24}\frac{\mathbf{k}^2}{4\mathcal{M}^2}\right)\frac{(M^2_{24}-M^2_{13})}{4M_{13}M_{24}}$$

$$-(g^V_{13}f^V_{24} - f^V_{13}g^V_{24})\sqrt{\frac{M_{13}M_{24}}{\mathcal{M}^2}}\Bigg\}/(M_{13}M_{24}) \tag{23}$$

c) scalar-meson exchange:

$$\Omega^{(S)}_1 = -g^S_{13}g^S_{24}\left(1 + \frac{\mathbf{k}^2}{8M_{13}M_{24}} - \frac{\mathbf{q}^2}{2M_{13}M_{24}}\right)$$

$$\Omega^{(S)}_4 = -g^S_{13}g^S_{24}\left(\frac{1}{2M_{13}M_{24}}\right)$$

$$\Omega^{(S)}_5 = \quad g^S_{13}g^S_{24}\left(\frac{1}{16M^2_{13}M^2_{24}}\right)$$

$$\Omega^{(S)}_6 = -g^S_{13}g^S_{24}\left(\frac{M^2_{24}-M^2_{13}}{4M^2_{13}M^2_{24}}\right) \tag{24}$$

d) 'diffractive-exchange':

The Ω^D_α are the same as for scalar-meson-exchange (24), but with $\pm g^S_{13}g^S_{24}$ replaced by $\mp g^D_{13}g^D_{24}$.

In the expressions for Ω^P, Ω^V, and Ω^S given above, M_{13} and M_{24} denote the average baryon masses, respectively $M_{13} = (M_1 + M_3)/2$ and $M_{24} = (M_2 + M_4)/2$, and m denotes the mass of the exchanged meson. In deriving these formulae for the Ω's there is used $1/M^2_N + 1/M^2_Y \approx 2/M_{24}M_{13}$, which holds to a very good approximation for NN and YN scattering.

In case of the strangeness carrying exchanges (K, K^*, κ, K^{**}) the rules for the modification of (22 - 24) have been given in [7,9]. In these cases one must make in (22-24) the substitutions $M_Y, M_N \longrightarrow (M_Y M_N)^{1/2}$ and because of the exchange character add an overall minus sign. In the case of the K^* one has furthermore to add the contribution of the second term of the vector-meson propagator, see [7], equation (26).

From the Ω_α's and writing

$$\mathbf{k}^2 = g^2_f + g^2_i - 2q_f g_i z \quad , \quad \mathbf{q}^2 = \frac{1}{4}(q^2_f + g^2_i + 2q_f g_i z) \tag{25}$$

where $z = \cos\theta$, one sees that the potentials can be written in the form

$$V_\alpha(\mathbf{k}^2,\mathbf{q}^2) = \Big(X_\alpha + zY_\alpha + z^2 Z_\alpha\Big)\cdot \Delta^{(X)}(\mathbf{k}^2,m^2,\Lambda^2)\ . \tag{26}$$

This holds, obviously, for the contributions of each of the different types of exchange separately. In the following we will work out the latter explicitly. For each type of exchange, the coefficients X_α, Y_α, and Z_α can readily be read off from the Ω_α's. The results for $X = P,\ V,\ S$, and D are listed in appendix A.

IV. PARTIAL WAVE ANALYSIS

The basic partial wave projections needed are

$$\begin{aligned} U_L(F,x) &= \frac{1}{2}\int_{-1}^{+1} dz \frac{P_L(z)F(z)}{x-z}\ , \\ R_L(F) &= \frac{1}{2}\int_{-1}^{+1} dz P_L(z)F(z)\ , \\ S_L(F) &= \frac{1}{2}\int_{-1}^{+1} dz\ z\ P_L(z)F(z)\ , \end{aligned} \tag{27}$$

where the form factor $F(z)$ and x are

$$F(z) = \exp\Big(-\mathbf{k}^2/\Lambda^2\Big)\quad,\quad x = \frac{q_f^2+g_i^2+m^2}{2q_f q_i}\ , \tag{28}$$

with m the mass of the exchanged boson. For $\Lambda\to\infty$, *i.e.* $F(z)\to 1$, the projections (27) become

$$U_L(1,z) = Q_L(x)\quad,\quad R_L(1) = \delta_{L\,0}\quad,\quad S_L(1) = \frac{1}{3}\delta_{L\,1}\ . \tag{29}$$

Writing

$$V(\mathbf{q}_f,\mathbf{q}_i) = \sum_{\alpha=1}^{6} V_\alpha(\mathbf{q}_f,\mathbf{q}_i)\,(\mathbf{q}_f|P_\alpha|\mathbf{q}_i)\ , \tag{30}$$

the partial wave expansion of the V_α-functions reads

$$V_\alpha(\mathbf{q}_f,\mathbf{q}_i) = \sum_{L=0}^{\infty}\ (2L+1)\ V_L^{(\alpha)}(x)\ P_L(\cos\theta)\ . \tag{31}$$

Using (20) and (26) the partial waves $V_L^{(\alpha)}(x)$ for $X = P,\ V,\ S$ become

$$V_L^{(\alpha)}(x) = \frac{1}{2q_i q_f}\Big[\Big(X_\alpha + xY_\alpha + x^2 Z_\alpha\Big)U_L - (Y_\alpha + xZ_\alpha)\,R_L\ - Z_\alpha S_L\Big] \tag{32}$$

and for $X = D$

$$V_L^{(\alpha)}(x) = \frac{1}{\mathcal{M}^2}\left[X_\alpha R_L + Y_\alpha S_L\right]\ . \tag{33}$$

In the last expression we have used the fact that in this case there does not appear a z^2-term in the potentials.

Distinguishing between the partial waves with parity $P = (-)^J$ and $P = -(-)^J$, we write the potential matrix elements on the LSJ-basis in the following way (see *e.g.* [13], section 7):
(i) $P = (-)^J$:

$$(q_f; L'S'J'M'|\ V\ |g_i; LSJM) = 4\pi\ V^{J,+}(S', S)\delta_{J'J}\ \delta_{M'M}\ \delta_{L'L}\ . \tag{34}$$

(ii) $P = -(-)^J$:

$$(q_f; L'S'J'M'|\ V\ |g_i; LSJM) = 4\pi\ \delta_{J'J}\ \delta_{M'M}\ \delta_{S'S}\ V^{J,-}(L', L)\ . \tag{35}$$

For notational convenience we will use as an index the parity factor η, which is defined by writing $P = \eta(-)^J$. The $P = (-)^J$-states contain the spin singlet and triplet-uncoupled states($\eta = +$), and the $P = -(-)^J$-states contain the spin triplet-coupled states ($\eta = -$).

In the soft-core model [4] the spin singlet-triplet transitions are neglected. This because in *NN* the mass differences are small and $g^V_{13}f^V_{24} - f^V_{13}g^V_{24} = 0$, one can neglect V_6 and hence the spin singlet-triplet transitions. However, for the hyperon-nucleon and cascade-nucleon channels this is not the case and these transitions can be significant, especially in hypernuclei [16]. Therefore we include the corresponding potentials in this work.

In appendix B the partial wave matrix elements of the operators P_α are evaluated in somewhat detail. Below we list the partial wave matrix elements for $\eta = \pm$ for the different $V^\alpha\ P_\alpha$, $(\alpha = 1, \ldots, 6)$. Here we restrict ourselves to the matrix elements $\neq 0$.
1. *central* $P_1 = 1$:

$$(q_f; L'S'J'M'|V^{(1)}P_1|g_i; LSJM) = 4\pi\ \delta_{J'J}\ \delta_{M'M}\ F_1^{J,\eta}(L'\ S', L\ S)\ , \tag{36}$$

$$\text{with} \quad F_1^{J,\eta}(L'\ S', L\ S)\ = \delta_{L'L}\ \delta_{S'S}\ V_L^{(1)}(x)$$

2. *spin-spin* $P_2 = \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$:

$$(q_f; L'S'J'M'|V^{(2)}P_2|g_i; LSJM) = 4\pi\ \delta_{J'J}\ \delta_{M'M}\ F_2^{J,\eta}(L'\ S', L\ S)\ , \tag{37}$$

$$\text{with} \quad F_2^{J,\eta}(L'\ S', L\ S)\ = \delta_{L'L}\ \delta_{S'S}\ [2S(S+1) - 3]\ V_L^{(2)}(x)$$

3. *tensor* $P_3 = (\boldsymbol{\sigma}_1 \cdot \mathbf{k})(\boldsymbol{\sigma}_2 \cdot \mathbf{k}) - \frac{1}{3}(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\mathbf{k}^2$:

$$(q_f; L'S'J'M'|V^{(3)}P_3|g_i; LSJM) = \frac{8\pi}{3}(q_f^2 + g_i^2)\ \delta_{J'J}\ \delta_{M'M}\ F_3^{J,\eta}(i, j)\ , \tag{38}$$

where $i = S'$ and $j = S$ for $\eta = +$, respectively $i = L'$ and $j = L$ for $\eta = -$.
(i) triplet uncoupled: $L = L' = J,\ S = S' = 1$

$$F_3^{J,+}(1,1) = \left[V_J^{(3)} - \frac{1}{2}\sin 2\psi\left(\frac{2J+3}{2J+1}V_{J-1}^{(3)} + \frac{2J-1}{2J+1}V_{J+1}^{(3)}\right)\right] \tag{39}$$

(ii) triplet coupled: $L = J \pm 1,\ L' = J \pm 1,\ S = S' = 1$

$$F_3^{J,-}(J-1,J-1) = \frac{J-1}{2J+1}\left[-V_{J-1}^{(3)} + \frac{1}{2}\sin 2\psi \cdot \right.$$
$$\left. \times \left\{\frac{2J-3}{2J-1}V_J^{(3)} + \frac{2J+1}{2J-1}V_{J-2}^{(3)}\right\}\right]$$
$$F_3^{J,-}(J-1,J+1) = -3\frac{\sqrt{J(J+1)}}{2J+1}\left[-\sin 2\psi\ V_J^{(3)} + \right.$$
$$\left. + \left(\cos^2\psi V_{J-1}^{(3)} + \sin^2\psi V_{J+1}^{(3)}\right)\right]$$
$$F_3^{J,-}(J+1,J-1) = -3\frac{\sqrt{J(J+1)}}{2J+1}\left[-\sin 2\psi\ V_J^{(3)} + \right.$$
$$\left. + \left(\sin^2\psi V_{J-1}^{(3)} + \cos^2\psi V_{J+1}^{(3)}\right)\right]$$
$$F_3^{J,-}(J+1,J+1) = \frac{J+2}{2J+1}\left[-V_{J+1}^{(3)} + \frac{1}{2}\sin 2\psi \cdot \right.$$
$$\left. \times \left\{\frac{2J+5}{2J+3}V_J^{(3)} + \frac{2J+1}{2J+3}V_{J+2}^{(3)}\right\}\right] \qquad (40)$$

where we introduced

$$\cos\psi = \frac{g_i}{\sqrt{q_f^2+g_i^2}} \quad , \quad \sin\psi = \frac{q_f}{\sqrt{q_f^2+g_i^2}} \qquad (41)$$

4. *spin-orbit* $P_4 = \frac{i}{2}(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)\cdot\mathbf{n}$:

$$(q_f; L'S'J'M'|V^{(4)}P_4|g_i; LSJM) = 4\pi\, q_f g_i \delta_{J'J}\,\delta_{M'M}\,F_4^{J,\eta}(i,j)\ . \qquad (42)$$

(i) triplet uncoupled: $L = L' = J,\ S = S' = 1$

$$F_4^{J,+}(1,1) = -\left(V_{J-1}^{(4)} - V_{J+1}^{(4)}\right)/(2J+1) \qquad (43)$$

(ii) triplet coupled: $L = J \pm 1,\ L' = J \pm 1,\ S = S' = 1$

$$F_4^{J,-}(J-1,J-1) = \frac{(J-1)}{(2J-1)}\left(V_{J-2}^{(4)} - V_J^{(4)}\right)$$
$$F_4^{J,-}(J+1,J+1) = -\frac{(J+2)}{(2J+3)}\left(V_J^{(4)} - V_{J+2}^{(4)}\right) \qquad (44)$$

5. *quadratic-spin-orbit* $P_5 = (\boldsymbol{\sigma}_1\cdot\mathbf{n})(\boldsymbol{\sigma}_2\cdot\mathbf{n})$:

$$(q_f; L'S'J'M'|V^{(5)}P_5|g_i; LSJM) = 4\pi q_f^2 g_i^2\ \delta_{J'J}\delta_{M'M}F_5^{J,\eta}(i,j) \qquad (45)$$

(i) singlet: $L = L' = J,\ S = S' = 0$

$$F_5^{J,+}(0,0) = e_{0,0}^{(5,+)}\ V_{J-2}^{(5)} + f_{0,0}^{(5,+)}\ V_J^{(5)} + g_{0,0}^{(5,+)}\ V_{J+2}^{(5)} \qquad (46)$$

(ii) triplet uncoupled: $L = L' = J,\ S = S' = 1$

$$F_5^{J,+}(1,1) = e_{1,1}^{(5,+)}\ V_{J-2}^{(5)} + f_{1,1}^{(5,+)}\ V_J^{(5)} + g_{1,1}^{(5,+)}\ V_{J+2}^{(5)} \tag{47}$$

(iv) triplet coupled:

$$\begin{aligned}
F_5^{J,-}(J-1,J-1) &= e_{J-1,J-1}^{(5,-)}\ V_{J-3}^{(5)} + f_{J-1,J-1}^{(5,-)}\ V_{J-1}^{(5)} + g_{J-1,J-1}^{(5,-)}\ V_{J+1}^{(5)} \\
F_5^{J,-}(J\pm 1,J\mp 1) &= -f_{J+1,J-1}^{(5,-)}\ \left[V_{J+1}^{(5)} - V_{J-1}^{(5)}\right] \\
F_5^{J,-}(J+1,J+1) &= e_{J+1,J+1}^{(5,-)}\ V_{J-1}^{(5)} + f_{J+1,J+1}^{(5,-)}\ V_{J+1}^{(5)} + g_{J+1,J+1}^{(5,-)}\ V_{J+3}^{(5)}
\end{aligned} \tag{48}$$

where the coefficients $e_{S',S}^{(5,+)}$ and $e_{L',L}^{(5,-)}$ etc. are given in appendix C.
6. *antisymmetric spin-orbit* $P_6 = \frac{i}{2}(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\cdot\mathbf{n}$:

$$(q_f; L'S'J'M'|V^{(6)}P_6|g_i; LSJM) = 4\pi\, q_f g_i \delta_{J'J}\,\delta_{M'M}\, F_6^{J,\eta}(S',S)\ . \tag{49}$$

(i) singlet-triplet uncoupled: $L = L' = J,\ S = 0,\ S' = 1$

$$F_6^{J,+}(1,0) = F_6^{J,+}(0,1)\ = \frac{\sqrt{J(J+1)}}{2J+1}\left(V_{J-1}^{(6)} - V_{J+1}^{(6)}\right)\ . \tag{50}$$

With the matrix elements of this section, the partial waves for the potentials can be readily derived. This will be done in the next sections for the pseudo-scalar, the vector, the scalar, and the diffractive potentials. Henceforth, we will use the following shorthand notation [11] for the potentials:
(i) $P = (-)^J$:

$$\begin{aligned}
V_{0,0}^J &= V^{J,+}(0,0) \quad , \quad V_{0,2}^J = V^{J,+}(0,1) \\
V_{2,0}^J &= V^{J,+}(1,0) \quad , \quad V_{2,2}^J = V^{J,+}(1,1)
\end{aligned} \tag{51}$$

(ii) $P = -(-)^J$:

$$\begin{aligned}
V_{1,1}^J &= V^{J,-}(J-1,J-1) \quad , \quad V_{1,3}^J = V^{J,-}(J-1,J+1) \\
V_{3,1}^J &= V^{J,-}(J+1,J-1) \quad , \quad V_{3,3}^J = V^{J,-}(J+1,J+1)
\end{aligned} \tag{52}$$

where it is always understood that the final and initial state momenta are respectively q_f and q_i. So $V_{0,0}^J = V_{0,0}^J(q_f,q_i)$ etc. Since

$$V_{2,0}^J(q_f,q_i) = V_{0,2}^J(q_i,q_f) \quad , \quad V_{3,1}^J(q_f,q_i) = V_{1,3}^J(q_i,q_f) \tag{53}$$

we will give in case of the off-diagonal terms only the explicit expressions for $V_{0,2}^J(q_f,q_i)$ and $V_{1,3}^J(q_f,q_i)$.

In the appendices E–H we have listed the explicit results for the pseudoscalar , vector, scalar, and diffractive momentum space potentials.

V. QUADRATIC SPIN-ORBIT POTENTIALS

In [4] and [7] the potential in configuration space has for the quadratic-spin-orbit the Q_{12}-operator. In going from momentum space to configuration space by the Fourier transformation, several non-local terms were neglected at this point. In order to reproduce the results of [4] and [7] exactly in momentum space, we must evaluate the inverse Fourier transformation for the Q_{12}-operator. This inverse Fourier transformation is carried through explicitly in appendix D. There it appears that upon Fourier transforming the soft-core potentials $V_Q(r)\ Q_{12}$ for the different exchanges one gets the result that

$$
\begin{aligned}
\widetilde{V}_Q(\mathbf{k},\mathbf{q}) &= V_5(\mathbf{k}^2)\ P_5 + \Delta\widetilde{V}_Q(\mathbf{k},\mathbf{q}) \\
\Delta\widetilde{V}_Q(\mathbf{k},\mathbf{q}) &= -\left\{2(\mathbf{S}\cdot\mathbf{q}_f)(\mathbf{S}\cdot\mathbf{q}_i) - i(\mathbf{q}_f\times\mathbf{q}_i)\cdot\mathbf{S}\right\}\widetilde{g}(\mathbf{k}^2) \\
&\quad + \{\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + 1\}\ \ (\mathbf{q}_f\cdot\mathbf{q}_i)\ \widetilde{g}(\mathbf{k}^2)\ ,
\end{aligned}
\tag{54}
$$

From appendix C it appears that the relation between $\widetilde{g}(\mathbf{k}^2)$ and $V_5(\mathbf{k}^2)$ is given by

$$
d\widetilde{g}(\mathbf{k}^2)/d\mathbf{k}^2 = \frac{1}{2}\ V_5(\mathbf{k}^2) = \frac{1}{2}\sum_X \Omega_5^{(X)}\Delta^{(X)}(\mathbf{k}^2, m^2, \Lambda^2)\ , \tag{55}
$$

since Ω_5 does not depend on $\mathbf{k}^2$. In order to obtain the exact momentum space potentials corresponding to the soft-core ones, we must include in addition to those given in the foregoing sections, also the contributions due to $\Delta\widetilde{V}_Q(\mathbf{k},\mathbf{q})$. With the results of appendix B and C, the partial wave projection of $\Delta V_Q(\mathbf{k},\mathbf{q})$ can be written down straightforwardly. We find
(i) *vector-meson*:

$$
\begin{aligned}
\Delta V^J_{0,0}(V) &= -4\pi\ q_f g_i\ X_Q^{(V)}\left\{\frac{2}{2J+1}\left(J\widetilde{g}^{(V)}_{J-1} + (J+1)\widetilde{g}^{(V)}_{J+1}\right)\right\}(F,x) \\
\Delta V^J_{2,2}(V) &= \ \ 4\pi\ q_f g_i\ X_Q^{(V)}\left\{\frac{1}{2J+1}\left(-\widetilde{g}^{(V)}_{J-1} + \widetilde{g}^{(V)}_{J+1}\right)\right\}(F,x) \\
\Delta V^J_{1,1}(V) &= \ \ 4\pi\ q_f g_i\ X_Q^{(V)}\left\{\frac{J-1}{2J-1}\widetilde{g}^{(V)}_{J-2} + \frac{2J^2-J+1}{(2J-1)(2J+1)}\widetilde{g}^{(V)}_J\right\}(F,x) \\
\Delta V^J_{1,3}(V) &= \ \ 4\pi\ q_f g_i\ X_Q^{(V)}\ \frac{2\sqrt{J(J+1)}}{2J+1}\ \widetilde{g}^{(V)}_J(F,x) \\
\Delta V^J_{3,3}(V) &= \ \ 4\pi\ q_f g_i\ X_Q^{(V)}\left\{\frac{(2J^2+5J+4)}{(2J+1)(2J+3)}\widetilde{g}^{(V)}_J + \frac{J+2}{2J+3}\widetilde{g}^{(V)}_{J+2}\right\}(F,x)
\end{aligned}
$$

(ii) *scalar-meson*:

$$
\begin{aligned}
\Delta V^J_{0,0}(S) &= -4\pi\ q_f g_i\ X_Q^{(S)}\left\{\frac{2}{2J+1}\left(J\widetilde{g}^{(S)}_{J-1} + (J+1)\widetilde{g}^{(S)}_{J+1}\right)\right\}(F,x) \\
\Delta V^J_{2,2}(S) &= \ \ 4\pi\ q_f g_i\ X_Q^{(S)}\left\{\frac{1}{2J+1}\left(-\widetilde{g}^{(S)}_{J-1} + \widetilde{g}^{(S)}_{J+1}\right)\right\}(F,x)
\end{aligned}
$$

$$\Delta V^J_{1,1}(S) = \quad 4\pi\ q_f g_i\ X_Q^{(S)} \left\{ \frac{J-1}{2J-1}\widetilde{g}^{(S)}_{J-2} + \frac{2J^2-J+1}{(2J-1)(2J+1)}\widetilde{g}^{(S)}_J \right\}(F,x)$$

$$\Delta V^J_{1,3}(S) = \quad 4\pi\ q_f g_i\ X_Q^{(S)}\ \frac{2\sqrt{J(J+1)}}{2J+1}\ \widetilde{g}^{(S)}_J(F,x)$$

$$\Delta V^J_{3,3}(S) = \quad 4\pi\ q_f g_i\ X_Q^{(S)} \left\{ \frac{(2J^2+5J+4)}{(2J+1)(2J+3)}\widetilde{g}^{(S)}_J + \frac{J+2}{2J+3}\widetilde{g}^{(S)}_{J+2} \right\}(F,x)$$

(iii) *diffractive*:

$$\Delta V^J_{0,0}(D) = -4\pi\ q_f g_i\ X_Q^{(D)} \left\{ \frac{2}{2J+1}\left(J\widetilde{g}^{(D)}_{J-1} + (J+1)\widetilde{g}^{(D)}_{J+1}\right) \right\}(F,x)$$

$$\Delta V^J_{2,2}(D) = \quad 4\pi\ q_f g_i\ X_Q^{(D)} \left\{ \frac{1}{2J+1}\left(-\widetilde{g}^{(D)}_{J-1} + \widetilde{g}^{(D)}_{J+1}\right) \right\}(F,x)$$

$$\Delta V^J_{1,1}(D) = \quad 4\pi\ q_f g_i\ X_Q^{(D)} \left\{ \frac{J-1}{2J-1}\widetilde{g}^{(D)}_{J-2} + \frac{(2J^2-J+1)}{(2J-1)(2J+1)}\widetilde{g}^{(D)}_J \right\}(F,x)$$

$$\Delta V^J_{1,3}(D) = \quad 4\pi\ q_f g_i\ X_Q^{(D)}\ \frac{2\sqrt{J(J+1)}}{2J+1}\ \widetilde{g}^{(D)}_J(F,x)$$

$$\Delta V^J_{3,3}(D) = \quad 4\pi\ q_f g_i X_Q^{(D)} \left\{ \frac{(2J^2+5J+4)}{(2J+1)(2J+3)}\widetilde{g}^{(D)}_J + \frac{J+2}{2J+3}\widetilde{g}^{(D)}_{J+2} \right\}(F,x)$$

The connection between $\widetilde{g}_J$ and the $V_J^{(Q)}$ is rather simple as can be seen from appendix C. In fact for $J \neq 0$ one has

$$\widetilde{g}_J(x) = \frac{q_f g_i}{2J+1}\left[\widetilde{h}_{J+1}(x) - \widetilde{h}_{J-1}(x)\right] \tag{56}$$

where $\widetilde{h}_{J\pm1}$ is given in (C8). Using this relation it is straightforward to check that the contributions from the quadratic spin-orbit operators to the sum $V_5^J(X)+\Delta V_5^J(X)$ vanishes. This is in accordance with the fact that the Q_{12}-operator has no off-diagonal matrix elements in configuration space.

VI. CONCLUSION AND RESULTS

The formulas given in this paper have been checked numerically in two steps. First we have constructed a momentum space program for plane waves using the formulas of section III. Doing the Fourier transformation to configuration space numerically we recovered the potentials of [4]. Then, we have computed the amplitudes $M_{SS}, M_{m'm}$ of [8] by the summation of the partial waves using the formulas of appendices E–H and section V. These checked with the same amplitudes as computed by the above mentioned plane wave momentum space program. Apart from this, Gibson and Stadler [17] have solved the partial wave Lippmann-Schwinger equation, using our computer code based on this paper, and reproduced the phase shifts of [4].

In Figures (1-11) we show the typical structure of the momentum space potentials in 3-dimensional plots and in the corresponding altitude charts the lowest partial wave nucleon-nucleon potentials in momentum space. Horizontally, g_i and q_f are plotted logarithmically in MeV from 0 to 10^5. The potentials are plotted on the vertical axis in units fm^2. These partial waves are the exact momentum space counterparts of the soft-core Nijmegen model [4].

Finally, we mention that computer programs are available on request.

APPENDIX A

(i) *Pseudo-scalar-meson exchange*:

$$\begin{aligned} X_\sigma^{(P)} &= -f_{13}^P f_{24}^P \left(\frac{q_f^2+g_i^2}{3m_S^2}\right) \quad , \quad Y_\sigma^{(P)} = f_{13}^P f_{24}^P \left(\frac{2q_f g_i}{3m_S^2}\right) \\ X_T^{(P)} &= -f_{13}^P f_{24}^P \left(\frac{1}{m_S^2}\right) \end{aligned} \tag{A1}$$

(ii) *Vector-meson exchange*:

$$\begin{aligned} X_C^{(V)} &= g_{13}^V g_{24}^V \left(1+\frac{q_f^2+g_i^2}{4M_{13}M_{24}}\right) - \left(g_{13}^V f_{24}^V \frac{M_{13}}{\mathcal{M}} + f_{13}^V g_{24}^V \frac{M_{24}}{\mathcal{M}}\right)\left(\frac{q_f^2+g_i^2}{4M_{13}M_{24}}\right) \\ &\quad + f_{13}^V f_{24}^V \frac{(q_f^2+g_i^2)^2}{16\mathcal{M}^2 M_{13}M_{24}} \\ Y_C^{(V)} &= g_{13}^V g_{24}^V \left(\frac{q_f g_i}{M_{13}M_{24}}\right) + \left(g_{13}^V f_{24}^V \frac{M_{13}}{\mathcal{M}} + f_{13}^V g_{24}^V \frac{M_{24}}{\mathcal{M}}\right)\left(\frac{q_f g_i}{2M_{13}M_{24}}\right) \\ &\quad - f_{13}^V f_{24}^V \frac{q_f g_i}{4\mathcal{M}^2}\left(\frac{q_f^2+g_i^2}{M_{13}M_{24}}\right) \\ Z_C^{(V)} &= f_{13}^V f_{24}^V \frac{q_f^2 g_i^2}{4\mathcal{M}^2 M_{13}M_{24}} \\ X_T^{(V)} &= \left\{\left(g_{13}^V + f_{13}^V \frac{M_{13}}{\mathcal{M}}\right)\left(g_{24}^V + f_{24}^V \frac{M_{24}}{\mathcal{M}}\right) - f_{13}^V f_{24}^V \frac{q_f^2+g_i^2}{8\mathcal{M}^2}\right\} / (4M_{13}M_{24}) \\ Y_T^{(V)} &= f_{13}^V f_{24}^V \frac{q_f g_i}{16\mathcal{M}^2 M_{13}M_{24}} \\ X_{SO}^{(V)} &= -\left\{12 g_{13}^V g_{24}^V + 8(g_{13}^V f_{24}^V + f_{13}^V g_{24}^V)\sqrt{\frac{M_{13}M_{24}}{\mathcal{M}^2}} - 3 f_{13}^V f_{24}^V \frac{q_f^2+g_i^2}{\mathcal{M}^2}\right\} / (8M_{13}M_{24}) \\ Y_{SO}^{(V)} &= -3 f_{13}^V f_{24}^V \frac{q_f g_i}{4M_{13}M_{24}\mathcal{M}^2} \end{aligned}$$

$$X_Q^{(V)} = -\left\{ g_{13}^V g_{24}^V + 4(g_{13}^V f_{24}^V + f_{13}^V g_{24}^V) \frac{\sqrt{M_{13}M_{24}}}{\mathcal{M}} + 8 f_{13}^V f_{24}^V \frac{M_{13}M_{24}}{\mathcal{M}^2} \right\} /(16 M_{13}^2 M_{24}^2)$$

$$X_{ASO}^{(V)} = -\left\{ \left(g_{13}^V g_{24}^V + f_{13}^V f_{24}^V \frac{q_f^2 + g_i^2}{4\mathcal{M}^2} \right) \left(\frac{M_{24}^2 - M_{13}^2}{4 M_{13} M_{24}} \right) \right.$$

$$\left. - \left(g_{13}^V f_{24}^V - f_{13}^V g_{24}^V \right) \sqrt{\frac{M_{13}M_{24}}{\mathcal{M}^2}} \right\} /(M_{13}M_{24})$$

$$Y_{ASO}^{(V)} = + f_{13}^V f_{24}^V \, \frac{q_f g_i}{2\mathcal{M}^2} \left(\frac{M_{24}^2 - M_{13}^2}{4 M_{13}^2 M_{24}^2} \right) \tag{A2}$$

The spin-spin coefficients are given in terms of the $X_T^{(V)}$ etc. as follows

$$X_\sigma^{(V)} = -\frac{2}{3} \left(q_f^2 + g_i^2 \right) X_T^{(V)} \quad , \quad Y_\sigma^{(V)} = \frac{4}{3} q_f g_i \, X_T^{(V)} - \frac{2}{3} \left(q_f^2 + g_i^2 \right) Y_T^{(V)}$$

$$Z_\sigma^{(V)} = \frac{4}{3} q_f g_i \, Y_T^{(V)} \tag{A3}$$

(iii) *Scalar-meson exchange*:

$$\begin{aligned}
X_C^{(S)} &= -g_{13}^S g_{24}^S &&, \; Y_C^{(S)} = g_{13}^S g_{24}^S \left(\frac{q_f g_i}{2 M_{13} M_{24}} \right) \\
X_{SO}^{(S)} &= -g_{13}^S g_{24}^S \left(\frac{1}{2 M_{13} M_{24}} \right) &&, \; X_Q^{(S)} = g_{13}^S g_{24}^S \left(\frac{1}{16 M_{13}^2 M_{24}^2} \right) \\
X_{ASO}^{(S)} &= -g_{13}^S g_{24}^S \left(\frac{M_{24}^2 - M_{13}^2}{4 M_{13}^2 M_{24}^2} \right)
\end{aligned} \tag{A4}$$

(iv) *Diffractive exchange*:

$$\begin{aligned}
X_C^{(D)} &= g_{13}^D g_{24}^D &&, \; Y_C^{(D)} = -g_{13}^D g_{24}^D \left(\frac{q_f g_i}{2 M_{13} M_{24}} \right) \\
X_{SO}^{(D)} &= g_{13}^D g_{24}^D \left(\frac{1}{2 M_{13} M_{24}} \right) &&, \; X_Q^{(D)} = -g_{13}^D g_{24}^D \left(\frac{1}{16 M_{13}^2 M_{24}^2} \right) \\
X_{ASO}^{(D)} &= g_{13}^D g_{24}^D \left(\frac{M_{24}^2 - M_{13}^2}{4 M_{13}^2 M_{24}^2} \right)
\end{aligned} \tag{A5}$$

APPENDIX B

The spherical wave functions in momentum space with quantum numbers J, L, S, are in the SYM-convention [8]

$$\mathcal{Y}_{JLS}^M(\hat{\mathbf{p}}) = i^L \, C^{J \; L \; S}_{M \; m \; \mu} Y_m^L(\hat{\mathbf{p}}) \chi_\mu^S \tag{B1}$$

where χ is the two-nucleon spin wave function [18]. Then

$$(\mathbf{S}\cdot\hat{\mathbf{p}})\,\mathcal{Y}^M_{JLS}(\hat{\mathbf{p}}) = -\sqrt{6}\; i\;(-)^L \left\{ \sqrt{\frac{L}{2L-1}} \begin{bmatrix} L & S & J \\ 1 & 1 & 0 \\ L-1 & S & J \end{bmatrix} \mathcal{Y}^M_{JL-1S}(\hat{\mathbf{p}}) \right.$$

$$\left. + \sqrt{\frac{L+1}{2L+3}} \begin{bmatrix} L & S & J \\ 1 & 1 & 0 \\ L+1 & S & J \end{bmatrix} \mathcal{Y}^M_{JL+1S}(\hat{\mathbf{p}}) \right\}$$

where the $9j$-symbols differ from [19], formula (6.4.4), in the replacement of the $3j$-symbols by the Clebsch-Gordan coefficients and by leaving out the m_{33}-summation (see [20]). Working this out explicitly, we find

$$\begin{aligned} (\mathbf{S}\cdot\hat{\mathbf{p}})\,\mathcal{Y}^M_{JJ-11}(\hat{\mathbf{p}}) &= -i\; a_J\; \mathcal{Y}^M_{JJ1}(\hat{\mathbf{p}}) \\ (\mathbf{S}\cdot\hat{\mathbf{p}})\,\mathcal{Y}^M_{JJ+11}(\hat{\mathbf{p}}) &= i\; b_J\; \mathcal{Y}^M_{JJ1}(\hat{\mathbf{p}}) \\ (\mathbf{S}\cdot\hat{\mathbf{p}})\,\mathcal{Y}^M_{J\;J\;1}(\hat{\mathbf{p}}) &= i\; a_J\; \mathcal{Y}^M_{JJ-11}(\hat{\mathbf{p}}) - i\; b_J\; \mathcal{Y}^M_{JJ+11}(\hat{\mathbf{p}})\;, \end{aligned} \tag{B2}$$

where

$$a_J = -\sqrt{\frac{J+1}{2J+1}} \qquad , \qquad b_J = -\sqrt{\frac{J}{2J+1}}\;. \tag{B3}$$

Ordering the states according to $L = J-1,\; L = J,\; L = J+1$, we can write in matrix form

$$\left(\begin{matrix} L\; =\; J-1 \\ J \\ J+1 \end{matrix} \middle\| \mathbf{S}\cdot\hat{\mathbf{p}} \middle\| \begin{matrix} L\; =\; J-1 \\ J \\ J+1 \end{matrix} \right) = \begin{pmatrix} 0 & ia_J & 0 \\ -ia_J & 0 & ib_J \\ 0 & -ib_J & 0 \end{pmatrix}\;. \tag{B4}$$

Similarly, using for $-i(\hat{\mathbf{q}}_f\times\hat{\mathbf{q}}_i)\cdot\mathbf{S}$ for sperical components the formula

$$-i(\hat{\mathbf{q}}_f\times\hat{\mathbf{q}}_i)_n = -\frac{4\pi}{3}\sqrt{2}\; C^{111}_{kln} Y^1_k(\hat{\mathbf{q}}_f) Y^1_l(\hat{\mathbf{q}}_i)\;, \tag{B5}$$

one can work out the partial wave matrix elements involving this operator.
From the results above one can derive the following useful partial wave projections for the spin triplet states:

$$(L'1J|V(\mathbf{k}^2)\,(\mathbf{S}\cdot\hat{\mathbf{q}}_i)^2\,|L1J) = 4\pi \begin{pmatrix} a_J^2 V_{J-1} & 0 & -a_J b_J V_{J-1} \\ 0 & V_J & 0 \\ -a_J b_J V_{J+1} & 0 & b_J^2 V_{J+1} \end{pmatrix}$$

$$(L'1J|(\mathbf{S}\cdot\hat{\mathbf{q}}_f)^2 V(\mathbf{k}^2)|L1J) = 4\pi \begin{pmatrix} a_J^2 V_{J-1} & 0 & -a_J b_J V_{J+1} \\ 0 & V_J & 0 \\ -a_J b_J V_{J-1} & 0 & b_J^2 V_{J+1} \end{pmatrix}$$

$$(L'1J|(\mathbf{S}\cdot\hat{\mathbf{q}}_f) V(\mathbf{k}^2)(\mathbf{S}\cdot\hat{\mathbf{q}}_i)|L1J) = 4\pi \begin{pmatrix} a_J^2 V_J & 0 & -a_J b_J V_J \\ 0 & a_J^2 V_{J-1} + b_J^2 V_{J+1} & 0 \\ -a_J b_J V_J & 0 & b_J^2 V_J \end{pmatrix}$$

and

$$
(L'1J| - i(\hat{\mathbf{q}}_f \times \hat{\mathbf{q}}_i)\cdot \mathbf{S}\, V(\mathbf{k}^2)|L1J) = \frac{4\pi}{2J+1}\left\{\begin{array}{cl} (J-1)\,(V_{J-2}-V_J) & ,\ L=L'=J-1 \\ -\,(V_{J-1}-V_{J+1}) & ,\ L=L'=J \\ -(J+2)\,(V_J-V_{J+2}) & ,\ L=L'=J+1 \end{array}\right. \tag{B6}
$$

Using the identity

$$
(\boldsymbol{\sigma}_1\cdot\mathbf{a})(\boldsymbol{\sigma}_2\cdot\mathbf{a}) = 2(\mathbf{S}\cdot\mathbf{a})^2 - \mathbf{a}^2\ , \tag{B7}
$$

the tensor and the quadratic spin-orbit operators can be written as

$$
\begin{aligned}
&\text{(i)} \qquad P_3 = (\boldsymbol{\sigma}_1\cdot\mathbf{k})(\boldsymbol{\sigma}_2\cdot\mathbf{k}) - \frac{1}{3}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2) = \frac{1}{3}\left[g_i^2\, S_{12}(\hat{\mathbf{q}}_i) + q_f^2\, S_{12}(\hat{\mathbf{q}}_f)\right] \\
&\qquad\qquad -4\,(\mathbf{S}\cdot\mathbf{q}_f)\,(\mathbf{S}\cdot\mathbf{q}_i) + 2i\,(\mathbf{q}_f\times\mathbf{q}_i)\cdot\mathbf{S} + \frac{4}{3}\,(\mathbf{q}_f\cdot\mathbf{q}_i)\,\mathbf{S}^2
\end{aligned} \tag{B8}
$$

$$
\begin{aligned}
&\text{(ii)} \qquad P_5 = [\boldsymbol{\sigma}_1\cdot\mathbf{k}\times\mathbf{q}]\,[\boldsymbol{\sigma}_2\cdot\mathbf{k}\times\mathbf{q}] = \left(2\mathbf{S}^2-1\right)(\mathbf{q}_f\times\mathbf{q}_i)^2 \\
&\qquad\qquad +2\left[(\mathbf{S}\cdot\mathbf{q}_f)\,(\mathbf{S}\cdot\mathbf{q}_i) - i\,(\mathbf{q}_f\times\mathbf{q}_i)\cdot\mathbf{S}\right](\mathbf{q}_f\cdot\mathbf{q}_i) \\
&\qquad\qquad -2q_f^2 g_i^2\left[(\mathbf{S}\cdot\hat{\mathbf{q}}_f)^2 + (\mathbf{S}\cdot\hat{\mathbf{q}}_i)^2\right]\ ,
\end{aligned} \tag{B9}
$$

where the momentum-space tensor-operator S_{12} is defined as

$$
S_{12}(\hat{\mathbf{p}}) = 3(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{p}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{p}}) - (\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\ . \tag{B10}
$$

For the evaluation partial wave projection of the P_5-operator we need in addition the matrix element

$$
\begin{aligned}
(L'S'J'|\mathbf{a}^2 V(\mathbf{k}^2)|LSJ) = 4\pi q_f^2 g_i^2 \delta_{J',J}\delta_{L',L}\delta_{S',S}\Bigg[&2\frac{L^2+L-1}{(2L-1)(2L+3)}V_L + \\
&-\frac{L(L-1)}{(2L-1)(2L+1)}V_{L-2} - \frac{(L+1)(L+2)}{(2L+1)(2L+3)}V_{L+2}\Bigg]
\end{aligned}
$$

where

$$
\mathbf{a}^2 = (\mathbf{q}_i\times\mathbf{q}_f)^2 = q_f^2 g_i^2\left(1-z^2\right) \tag{B11}
$$

From the formulas given in this appendix the partial wave projections of the several potential forms, as given in appendices E–H can be derived in a straightforward manner. In case of an extra factor $(\mathbf{q}_f\cdot\mathbf{q}_i)$, as occurs for example in the second line of (B9), we simply use the expansion

$$
(\mathbf{q}_f\cdot\mathbf{q}_i)\,V(\mathbf{k}^2) = q_f g_i \sum_{L=0}^{\infty}(2L+1)\widetilde{V}_L(x)P_L(\cos\theta) \tag{B12}
$$

where

$$
\widetilde{V}_L = \frac{1}{2L+1}\left[(L+1)V_{L+1} + LV_{L-1}\right]\ . \tag{B13}
$$

APPENDIX C

In this appendix we derive the inverse Fourier transformation for the Q_{12}-operator. Starting from

$$\widetilde{V}_Q(\mathbf{k},\mathbf{q}) = \int d^3r' \int d^3r\, e^{i\mathbf{p}'\cdot\mathbf{r}'} V(\mathbf{r}',\mathbf{r})_Q\, e^{-i\mathbf{p}\cdot\mathbf{r}} \ , \tag{C1}$$

with the local configuration space potential

$$\begin{aligned} V_Q(\mathbf{r}',\mathbf{r}) &= \delta^3(\mathbf{r}'-\mathbf{r})\ f(r)\ Q_{12} \\ Q_{12} &= \frac{1}{2}\left(\boldsymbol{\sigma}_1\cdot\mathbf{L}\ \boldsymbol{\sigma}_2\cdot\mathbf{L} + \boldsymbol{\sigma}_2\cdot\mathbf{L}\ \boldsymbol{\sigma}_1\cdot\mathbf{L}\right) \ , \end{aligned} \tag{C2}$$

and using

$$r_i f(r) = -\nabla_i g(r) \quad , \quad r_i r_j f(r) = \left[-\nabla_i\nabla_j + \delta_{ij}\left(\frac{1}{r}\frac{d}{dr}\right)\right] h(r) \ , \tag{C3}$$

one finds upon carrying through the Fourier transformation (C1)

$$\begin{aligned} \widetilde{V}_Q(\mathbf{k},\mathbf{q}) = {} & [\boldsymbol{\sigma}_1\cdot\mathbf{q}\times\mathbf{k}]\,[\boldsymbol{\sigma}_2\cdot\mathbf{q}\times\mathbf{k}]\,\widetilde{h}(\mathbf{k}^2) - \left\{\left(\frac{1}{4}\mathbf{k}^2-\mathbf{q}^2\right)(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2) + \right. \\ & \left. + \left[(\boldsymbol{\sigma}_1\cdot\mathbf{q})\,(\boldsymbol{\sigma}_2\cdot\mathbf{q}) - \frac{1}{4}\,(\boldsymbol{\sigma}_1\cdot\mathbf{k})\,(\boldsymbol{\sigma}_2\cdot\mathbf{k})\right]\right\}\widetilde{g}(\mathbf{k}^2) \\ \equiv {} & \widetilde{h}(\mathbf{k}^2)P_5 + \Delta\widetilde{V}_Q(\mathbf{k},\mathbf{q}) \ , \end{aligned} \tag{C4}$$

where $\widetilde{h}(\mathbf{k}^2)$ and $\widetilde{g}(\mathbf{k}^2)$ are the Fourier transforms of respectively $h(r)$ and $g(r)$. Basically, *i.e.* apart from coupling constants etc.,

$$h(r) = \left[\frac{m}{4\pi}\phi^0_C(r)\right] \quad , \quad g(r) = \left(\frac{1}{r}\frac{d}{dr}\right)\left[\frac{m}{4\pi}\phi^0_C(r)\right] \ , \tag{C5}$$

In that case we have

$$\widetilde{h}(\mathbf{k}^2) = \Delta^{(X)}(\mathbf{k}^2, m^2, \Lambda^2) \tag{C6}$$

where $X = P,\ V$, or S. The function $\Delta^{(X)}$ is given in (20). From (C5) one can derive that $d\widetilde{g}(\mathbf{k}^2)/d\mathbf{k}^2 = (1/2)\ \widetilde{h}(\mathbf{k}^2)$, which leads to the Fourier transforms

$$\widetilde{g}(\mathbf{k}^2) = \begin{cases} -\frac{1}{2}\exp\left(m^2/\Lambda^2\right)\ E_1\left[(\mathbf{k}^2+m^2)/\Lambda^2\right] \ , \ (X = P,\ V,\ S) \\ -(2m_P^2/\mathcal{M}^2)\exp(-\mathbf{k}^2/4m_P^2) \qquad , \ (X = D) \ , \end{cases} \tag{C7}$$

where E_1 is the exponential integral [21]. The partial wave projection of $\widetilde{h}(\mathbf{k}^2)$ is

$$\widetilde{h}_J(x) = \begin{cases} (1/2q_f g_i)\ U_J(F,x) \ , \ (X = P,\ V,\ S) \\ (1/\mathcal{M}^2)R_J(F_D) \qquad , \ (X = D) \ . \end{cases} \tag{C8}$$

The partial wave projection of $\widetilde{g}(\mathbf{k}^2)$ can be shown to be

$$\widetilde{g}_J(x) = \frac{q_f g_i}{2J+1}\left[\widetilde{h}_{J+1}(x) - \widetilde{h}_{J-1}(x)\right] + \widetilde{g}(x, z=-1)\delta_{J0} \ , \tag{C9}$$

where for $J = 0$ it is understood that $\widetilde{h}_{-1} = \widetilde{h}_0$.

To facilitate the partial wave projection, we rewrite $\Delta\widetilde{V}_Q(\mathbf{k}, \mathbf{q})$ in terms of the total spin operator $\mathbf{S}$. After a little algebra we get

$$\begin{aligned}\Delta\widetilde{V}_Q(\mathbf{k}, \mathbf{q}) \ &= \ -\left\{2(\mathbf{S}\cdot\mathbf{q}_f)(\mathbf{S}\cdot\mathbf{q}_i) - i(\mathbf{q}_f\times\mathbf{q}_i)\cdot\mathbf{S}\right\}\widetilde{g}(\mathbf{k}^2) \\ &\quad + \left\{\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + 1\right\}\ (\mathbf{q}_f\cdot\mathbf{q}_i)\ \widetilde{g}(\mathbf{k}^2) \ .\end{aligned} \tag{C10}$$

Using now the results of appendix B, the partial wave projection of $\Delta V_Q(\mathbf{k}, \mathbf{q})$ can readily be obtained.

APPENDIX D

Here we give the coefficients for the partial wave projection of the quadratic-spin-orbit operator.

(i) singlet and triplet uncoupled:

$$\begin{aligned}F_5^{J,+}(0,0) &= e_{0,0}^{(5,+)}\ V_{J-2}^{(5)} + f_{0,0}^{(5,+)}\ V_J^{(5)} + g_{0,0}^{(5,+)}\ V_{J+2}^{(5)} \\ F_5^{J,+}(1,1) &= e_{1,1}^{(5,+)}\ V_{J-2}^{(5)} + f_{1,1}^{(5,+)}\ V_J^{(5)} + g_{1,1}^{(5,+)}\ V_{J+2}^{(5)} \ ,\end{aligned} \tag{D1}$$

where

$$\begin{aligned}e_{0,0}^{(5,+)} &= +\frac{J\ (J-1)}{(2J-1)\ (2J+1)}, e_{1,1}^{(5,+)} = +\frac{(J-1)\ (J+2)}{(2J-1)\ (2J+1)} \\ f_{0,0}^{(5,+)} &= -\frac{2(J^2+J-1)}{(2J-1)\ (2J+3)}, f_{1,1}^{(5,+)} = -\frac{2(J-1)\ (J+2)}{(2J-1)\ (2J+3)} \\ g_{0,0}^{(5,+)} &= +\frac{(J+1)\ (J+2)}{(2J+1)\ (2J+3)}, g_{1,1}^{(5,+)} = +\frac{(J-1)\ (J+2)}{(2J+1)\ (2J+3)} \ .\end{aligned} \tag{D2}$$

(ii) triplet coupled:

$$\begin{aligned}F_5^{J,-}(J-1, J-1) &= e_{J-1,J-1}^{(5,-)}\ V_{J-3}^{(5)} + f_{J-1,J-1}^{(5,-)}\ V_{J-1}^{(5)} + g_{J-1,J-1}^{(5,-)}\ V_{J+1}^{(5)} \\ F_5^{J,-}(J\pm 1, J\mp 1) &= -f_{J+1,J-1}^{(5,-)}\ \left[V_{J+1}^{(5)} - V_{J-1}^{(5)}\right] \\ F_5^{J,-}(J+1, J+1) &= e_{J+1,J+1}^{(5,-)}\ V_{J-1}^{(5)} + f_{J+1,J+1}^{(5,-)}\ V_{J+1}^{(5)} + g_{J+1,J+1}^{(5,-)}\ V_{J+3}^{(5)}\end{aligned} \tag{D3}$$

where

$$e^{(5,-)}_{J-1,J-1} = -\frac{(J-1)\ (J-2)}{(2J-1)\ (2J-3)} \qquad , e^{(5,-)}_{J+1,J+1} = -\frac{J\ (2J^2+7J+7)}{(2J+1)^2\ (2J+3)}\ ,$$

$$g^{(5,-)}_{J-1,J-1} = -\frac{(2J^2-3J+2)(J+1)}{(2J-1)\ (2J+1)^2}, g^{(5,-)}_{J+1,J+1} = -\frac{(J+2)\ (J+3)}{(2J+3)\ (2J+5)}\ , \tag{D4}$$

$$f^{(5,-)}_{J-1,J-1} = 2\frac{(2J^3-3J^2-2J+2)}{(2J+1)^2\ (2J-3)}\ ,$$

$$f^{(5,-)}_{J+1,J-1} = 2\frac{\sqrt{J(J+1)}}{(2J+1)^2}\ ,$$

$$f^{(5,-)}_{J+1,J+1} = 2\frac{(2J^3+9J^2+10J+1)}{(2J+1)^2\ (2J+5)}\ . \tag{D5}$$

APPENDIX E: PSEUDO-SCALAR-MESON POTENTIALS

With the coefficients $X^{(P)}_\alpha$ and $Y^{(P)}_\alpha$ of appendix A, the basic partial wave projections are

$$\begin{aligned} V^{(\sigma)}_L(x) &= \frac{1}{2g_i q_f}\left[\left(X^{(P)}_\sigma + xY^{(P)}_\sigma\right) U_L(F,x) - Y^{(P)}_\sigma\ R_L(F)\right] \\ V^{(T)}_L(x) &= \frac{1}{2g_i q_f}\ X^{(P)}_T\ U_L(F,x) \end{aligned} \tag{E1}$$

The momentum space partial wave potentials are

$$V^J_{0,0}(P) = -12\pi V^{(\sigma)}_J$$

$$\begin{aligned} V^J_{2,2}(P) = \quad & 4\pi\left[V^{(\sigma)}_J + \frac{2}{3}(q_f^2+g_i^2)\left\{V^{(T)}_J + \right.\right. \\ & \left.\left. -\frac{1}{2}\sin 2\psi\left(\frac{2J+3}{2J+1}V^{(T)}_{J-1} + \frac{2J-1}{2J+1}V^{(T)}_{J+1}\right)\right\}\right] \end{aligned}$$

$$\begin{aligned} V^J_{1,1}(P) = \quad & 4\pi\left[V^{(\sigma)}_{J-1} + \frac{2}{3}(q_f^2+g_i^2)\ \frac{J-1}{2J+1}\left\{-V^{(T)}_{J-1} + \right.\right. \\ & \left.\left. +\frac{1}{2}\sin 2\psi\left(\frac{2J-3}{2J-1}V^{(T)}_J + \frac{2J+1}{2J-1}V^{(T)}_{J-2}\right)\right\}\right] \end{aligned}$$

$$\begin{aligned} V^J_{1,3}(P) = & -8\pi\ (q_f^2+g_i^2)\ \frac{\sqrt{J(J+1}}{2J+1}\left[-\sin 2\psi\ V^{(T)}_J + \right. \\ & \left. +\left(\cos^2\psi\ V^{(T)}_{J-1} + \sin^2\psi\ V^{(T)}_{J+1}\right)\right] \end{aligned}$$

$$
\begin{aligned}
V^J_{3,3}(P) = \quad & 4\pi\left[V^{(\sigma)}_{J+1} + \frac{2}{3}(q_f^2 + g_i^2)\ \frac{J+2}{2J+1}\left\{-V^{(T)}_{J+1} + \frac{1}{2}\sin 2\psi\cdot\right.\right. \\
& \left.\left. \times\left(\frac{2J+5}{2J+3}V^{(T)}_J + \frac{2J+1}{2J+3}V^{(T)}_{J+2}\right)\right\}\right] \qquad \text{(E2)}
\end{aligned}
$$

APPENDIX F: VECTOR-MESON POTENTIALS

With the coefficients $X^{(V)}_\alpha$, $Y^{(V)}_\alpha$, and $Z^{(V)}_\alpha$ of appendix A, the basic partial wave projections are

$$
\begin{aligned}
V^{(C)}_L(x) \quad &= \frac{1}{2g_iq_f}\left[\left(X^{(V)}_C + xY^{(V)}_C + x^2Z^{(V)}_C\right)U_L(F,x) - \left(Y^{(V)}_C + xZ^{(V)}_C\right)\cdot\right. \\
&\left. \times R_L(F) - Z^{(V)}_C S_L(F)\right] \\
V^{(\sigma)}_L(x) \quad &= \frac{1}{2g_iq_f}\left[\left(X^{(V)}_\sigma + xY^{(V)}_\sigma + x^2Z^{(V)}_\sigma\right)U_L(F,x) - \left(Y^{(V)}_\sigma + xZ^{(V)}_\sigma\right)\cdot\right. \\
&\left. \times R_L(F) - Z^{(V)}_\sigma S_L(F)\right] \\
V^{(T)}_L(x) \quad &= \frac{1}{2g_iq_f}\left[\left(X^{(V)}_T + xY^{(V)}_T\right)U_L(F,x) - Y^{(V)}_T R_L(F)\right] \\
V^{(SO)}_L(x) \quad &= \frac{1}{2g_iq_f}\left[\left(X^{(V)}_{SO} + xY^{(V)}_{SO}\right)U_L(F,x) - Y^{(V)}_{SO} R_L(F)\right] \\
V^{(Q)}_L(x) \quad &= \frac{1}{2g_iq_f}\ X^{(V)}_Q\ U_L(F,x) \\
V^{(ASO)}_L(x) &= \frac{1}{2g_iq_f}\left[\left(X^{(V)}_{ASO} + xY^{(V)}_{ASO}\right)U_L(F,x) - Y^{(V)}_{ASO} R_L(F)\right] \qquad \text{(F1)}
\end{aligned}
$$

The momentum space partial wave potentials are

$$
\begin{aligned}
V^J_{0,0}(V) &= 4\pi\left[\left(V^{(C)}_J - 3V^{(\sigma)}_J\right) + q_f^2g_i^2\left(e^{(5,+)}_{0,0}\ V^{(Q)}_{J-2} + f^{(5,+)}_{0,0}\ V^{(Q)}_J + g^{(5,+)}_{0,0}\ V^{(Q)}_{J+2}\right)\right] \\
V^J_{0,2}(V) &= 4\pi q_f g_i \frac{\sqrt{J(J+1)}}{2J+1}\left(V^{(ASO)}_{J-1} - V^{(ASO)}_{J+1}\right) \\
V^J_{2,2}(V) &= 4\pi\left[\left(V^{(C)}_J +\ V^{(\sigma)}_J\right) + \frac{2}{3}(q_f^2 + g_i^2)\cdot\right. \\
&\quad \times\left\{V^{(T)}_J - \frac{1}{2}\sin 2\psi\left(\frac{2J+3}{2J+1}V^{(T)}_{J-1} + \frac{2J-1}{2J+1}V^{(T)}_{J+1}\right)\right\} \\
&\quad -q_f g_i\left(V^{(SO)}_{J-1} - V^{(SO)}_{J+1}\right)/(2J+1) \\
&\quad \left. +q_f^2g_i^2\left(e^{(5,+)}_{1,1}\ V^{(Q)}_{J-2} + f^{(5,+)}_{1,1}\ V^{(Q)}_J + g^{(5,+)}_{1,1}\ V^{(Q)}_{J+2}\right)\right]
\end{aligned}
$$

$$V^J_{1,1}(V) = 4\pi \left[\left(V^{(C)}_{J-1} + \ V^{(\sigma)}_{J-1}\right) + \frac{2}{3}(q_f^2 + g_i^2)\ \frac{J-1}{2J+1}\left\{-V^{(T)}_{J-1} + \right.\right.$$

$$+ \ \frac{1}{2}\sin 2\psi \left(\frac{2J-3}{2J-1}V^{(T)}_J + \frac{2J+1}{2J-1}V^{(T)}_{J-2}\right)\Big\}$$

$$+q_f g_i \ (J-1)\left(V^{(SO)}_{J-2} - V^{(SO)}_J\right)/(2J-1)$$

$$\left.+q_f^2 g_i^2 \left(e^{(5,-)}_{J-1,J-1}\ V^{(Q)}_{J-3} + f^{(5,-)}_{J-1,J-1}\ V^{(Q)}_{J-1} + g^{(5,-)}_{J-1,J-1}\ V^{(Q)}_{J+1}\right)\right]$$

$$V^J_{1,3}(V) = -4\pi \ \left[2(q_f^2 + g_i^2)\ \frac{\sqrt{J(J+1}}{2J+1}\left\{-\sin 2\psi V^{(T)}_J + \right.\right.$$

$$+ \left(\cos^2\psi \ V^{(T)}_{J-1} + \sin^2\psi \ V^{(T)}_{J+1}\right)\Big\}$$

$$\left.+q_f^2 g_i^2 f^{(5,-)}_{J+1,J-1}\ \left(V^{(Q)}_{J+1} - V^{(Q)}_{J-1}\right)\right]$$

$$V^J_{3,3}(V) = 4\pi \left[\left(V^{(C)}_{J+1} + \ V^{(\sigma)}_{J+1}\right) + \frac{2}{3}(q_f^2 + g_i^2)\ \frac{J+2}{2J+1}\left\{-V^{(T)}_{J+1} + \right.\right.$$

$$+\frac{1}{2}\sin 2\psi \left(\frac{2J+5}{2J+3}V^{(T)}_J + \frac{2J+1}{2J+3}V^{(T)}_{J+2}\right)\Big\}$$

$$-q_f g_i \ (J+2)\left(V^{(SO)}_J - V^{(SO)}_{J+2}\right)/(2J+3)$$

$$\left.+q_f^2 g_i^2 \left(e^{(5,-)}_{J+1,J+1}\ V^{(Q)}_{J-1} + f^{(5,-)}_{J+1,J+1}\ V^{(Q)}_{J+1} + g^{(5,-)}_{J+1,J+1}\ V^{(Q)}_{J+3}\right)\right] \quad \text{(F2)}$$

APPENDIX G: SCALAR-MESON POTENTIALS

With the coefficients $X^{(S)}_\alpha$ and $Y^{(S)}_\alpha$ of appendix A, the basic partial wave projections are

$$\begin{aligned}
V^{(C)}_L(x) &= \frac{1}{2g_i q_f}\left[\left(X^{(S)}_C + xY^{(S)}_C\right)U_L(F,x) - Y^{(S)}_C \ R_L(F)\right] \\
V^{(SO)}_L(x) &= \frac{1}{2g_i q_f}\ X^{(S)}_{SO}\ U_L(F,x) \\
V^{(Q)}_L(x) &= \frac{1}{2g_i q_f}\ X^{(S)}_Q\ U_L(F,x) \\
V^{(ASO)}_L(x) &= \frac{1}{2g_i q_f}\ X^{(S)}_{ASO}\ U_L(F,x)
\end{aligned} \quad \text{(G1)}$$

The momentum space partial wave potentials are

$$V^J_{0,0}(S) = 4\pi\left[V^{(C)}_J + q_f^2 g_i^2\left(e^{(5,+)}_{0,0}\ V^{(Q)}_{J-2} + f^{(5,+)}_{0,0}\ V^{(Q)}_J + g^{(5,+)}_{0,0}\ V^{(Q)}_{J+2}\right)\right]$$

$$V^J_{0,2}(S) = 4\pi q_f g_i \frac{\sqrt{J(J+1)}}{2J+1} \left(V^{(ASO)}_{J-1} - V^{(ASO)}_{J+1}\right)$$

$$V^J_{2,2}(S) = 4\pi \Big[V^{(C)}_J - q_f g_i \left(V^{(SO)}_{J-1} - V^{(SO)}_{J+1}\right)/(2J+1)$$
$$+q_f^2 g_i^2 \left(e^{(5,+)}_{1,1}\ V^{(Q)}_{J-2} + f^{(5,+)}_{1,1}\ V^{(Q)}_J + g^{(5,+)}_{1,1}\ V^{(Q)}_{J+2}\right)\Big]$$

$$V^J_{1,1}(S) = 4\pi \Big[V^{(C)}_{J-1} + q_f g_i\ (J-1) \left(V^{(SO)}_{J-2} - V^{(SO)}_J\right)/(2J-1)$$
$$+q_f^2 g_i^2 \left(e^{(5,-)}_{J-1,J-1}\ V^{(Q)}_{J-3} + f^{(5,-)}_{J-1,J-1}\ V^{(Q)}_{J-1} + g^{(5,-)}_{J-1,J-1}\ V^{(Q)}_{J+1}\right)\Big]$$

$$V^J_{1,3}(S) = -4\pi q_f^2 g_i^2 f^{(5,-)}_{J+1,J-1}\ \left(V^{(Q)}_{J+1} - V^{(Q)}_{J-1}\right)$$

$$V^J_{3,3}(S) = 4\pi \Big[V^{(C)}_{J+1} - q_f g_i\ (J+2) \left(V^{(SO)}_J - V^{(SO)}_{J+2}\right)/(2J+3)$$
$$+q_f^2 g_i^2 \left(e^{(5,-)}_{J+1,J+1}\ V^{(Q)}_{J-1} + f^{(5,-)}_{J+1,J+1}\ V^{(Q)}_{J+1} + g^{(5,-)}_{J+1,J+1}\ V^{(Q)}_{J+3}\right)\Big] \quad \text{(G2)}$$

APPENDIX H: DIFFRACTIVE POTENTIALS

With the coefficients $X^{(S)}_\alpha$ and $Y^{(S)}_\alpha$ of appendix A, the basic partial wave projections are

$$\begin{aligned}
V^{(C)}_L(x) &= \frac{1}{\mathcal{M}^2} \left[X^{(D)}_C\ R_L(F) + Y^{(D)}_C\ S_L(F)\right] \\
V^{(SO)}_L(x) &= \frac{1}{\mathcal{M}^2}\ X^{(D)}_{SO}\ R_L(F) \\
V^{(Q)}_L(x) &= \frac{1}{\mathcal{M}^2}\ X^{(D)}_Q\ R_L(F) \\
V^{(ASO)}_L(x) &= \frac{1}{\mathcal{M}^2}\ X^{(D)}_{ASO}\ R_L(F)
\end{aligned} \quad \text{(H1)}$$

The momentum space partial wave potentials are

$$V^J_{0,0}(D) = 4\pi \left[V^{(C)}_J + q_f^2 g_i^2 \left(e^{(5,+)}_{0,0}\ V^{(Q)}_{J-2} + f^{(5,+)}_{0,0}\ V^{(Q)}_J + g^{(5,+)}_{0,0}\ V^{(Q)}_{J+2}\right)\right]$$

$$V^J_{0,2}(D) = 4\pi q_f g_i \left(V^{(ASO)}_{J-1} - V^{(ASO)}_{J+1}\right)/(2J+1)$$

$$V^J_{2,2}(D) = 4\pi \Big[V^{(C)}_J - q_f g_i \left(V^{(SO)}_{J-1} - V^{(SO)}_J\right)/(2J-1)$$
$$+q_f^2 g_i^2 \left(e^{(5,+)}_{1,1}\ V^{(Q)}_{J-2} + f^{(5,+)}_{1,1}\ V^{(Q)}_J + g^{(5,+)}_{1,1}\ V^{(Q)}_{J+2}\right)\Big]$$

$$V^J_{1,1}(D) = 4\pi \Big[V^{(C)}_{J-1} + q_f g_i\ (J-1) \left(V^{(SO)}_{J-2} - V^{(SO)}_J\right)$$

$$+q_f^2 g_i^2 \left(e^{(5,-)}_{J-1,J-1}\ V^{(Q)}_{J-3} + f^{(5,-)}_{J-1,J-1}\ V^{(Q)}_{J-1} + g^{(5,-)}_{J-1,J-1}\ V^{(Q)}_{J+1}\right)\Big]$$

$$V^J_{1,3}(D) = -4\pi q_f^2 g_i^2 f^{(5,-)}_{J+1,J-1}\ \left(V^{(Q)}_{J+1} - V^{(Q)}_{J-1}\right)$$

$$\begin{aligned} V^J_{3,3}(D) = 4\pi \Big[& V^{(C)}_{J+1} - q_f g_i\ (J+2)\left(V^{(SO)}_J - V^{(SO)}_{J+2}\right)/(2J+3) \\ & +q_f^2 g_i^2 \left(e^{(5,-)}_{J+1,J+1}\ V^{(Q)}_{J-1} + f^{(5,-)}_{J+1,J+1}\ V^{(Q)}_{J+1} + g^{(5,-)}_{J+1,J+1}\ V^{(Q)}_{J+3}\right)\Big] \end{aligned} \tag{H2}$$

REFERENCES

[1] P.S. Signell and R.E. Marshak, Phys. Rev. **106**, 832 (1957); **109**, 1229 (1958).

[2] J.R. Bergervoet, P.C. van Campen, R.A.M. Klomp, J.-L. de Kok, Th.A. Rijken, V.G.J. Stoks, and J.J. de Swart, Phys. Rev. C**41**, 1435 (1990).

[3] V.G.J. Stoks, R.A.M. Klomp, M.C.M. Rentmeester, and J.J. de Swart, Phys. Rev. C**48**, 792 (1993).

[4] M.M. Nagels, T.A. Rijken, and J.J. de Swart, Phys. Rev. D**17**, 768 (1978).

[5] R.V. Reid, Jr., Ann. Phys. (N.Y.) **50**, 411 (1968).

[6] V.G.J. Stoks, R.A.M. Klomp, C.P.F. Terheggen, and J.J. de Swart, to be published in Phys. Rev. C (1994).

[7] P.M.M. Maessen, T.A. Rijken, and J.J. de Swart, Phys. Rev. D**40**, 2226 (1989).

[8] H.P. Stapp, T. Ypsilantis, and M. Metropolis, Phys. Rev. **105**, 311 (1957).

[9] M.M. Nagels, T.A. Rijken, and J.J. de Swart, Phys. Rev. D**15**, 2547 (1977).

[10] A.A. Logunov and A.N. Tavkhelidze, Nuovo Cimento **29**, 380 (1963);
R. Blankenbecler and R. Sugar, Phys. Rev. **142**, 1051 (1966);
R.H. Thompson, Phys. Rev. **D1**, 110 (1970);
M.H. Partovi and E.L. Lomon, Phys. Rev. **D2**, 1999 (1970).

[11] T.A. Rijken, Ann. Phys. (NY) **164**, 1 and 23 (1985).

[12] A. Gersten, P.A. Verhoeven, and J.J. de Swart, Nuovo Cimento **A26**, 375 (1975); P.A. Verhoeven, Ph.D. Thesis, University of Nijmegen (1976, unpublished).

[13] J.J. de Swart, M.M. Nagels, T.A. Rijken, and P.A. Verhoeven, Springer tracts in Modern Physics, Vol. **60**, 137 (1971).

[14] S. Okubo and R.E. Marshak, Ann. Phys. (NY) **4**, 166 (1958).

[15] In the NN-model [4] a small symmetry breaking was introduced by using $1/M_{13}M_{24}$ instead of $1/\mathcal{M}^2$ in $\Delta^D(\mathbf{k}^2, m^2, \Lambda^2)$. More recently however, it appeared that a large symmetry breaking is not likely at this point, see: P. Povh and J. Hufner, Phys. Rev. Lett. **58**, 1612 (1987).

[16] C.B. Dover and A. Gal, in *Progress in Particle and Nuclear Physics*, edited by D. Wilkinson (Pergamon, New York, 1984), Vol. 12, p. 171.

[17] B.F. Gibson and A. Stadler, private communication, 1991.

[18] In the SYM-convention [8] the configuration space basic JLS-states are $\mathcal{Y}^M_{JLS}(\hat{\mathbf{r}}) = C^{J}_{M}\,{}^{L}_{m}\,{}^{S}_{\mu} Y^L_m(\hat{\mathbf{r}})\chi^S_\mu$. Transformation to momentum space gives (B1). See *e.g.* J.R. Taylor, *Scattering Theory: The Quantum Theory on Nonrelativistic Collisions* (John Wiley & Sons, Inc., New York (1972)).

[19] A.R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton (1957)).

[20] The explicit relation between our $9j$-symbols and those of [19] eq. (6.4.4) is

$$\begin{bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{bmatrix} = (-)^{j_{31}+j_{13}-j_{32}-j_{23}} \; \left[(2j_{13}+1)(2j_{31}+1)(2j_{23}+1)(2j_{32}+1)\right]^{1/2}$$

$$\times \left\{ \begin{array}{ccc} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{array} \right\}$$

[21] M. Abramowitz and I.A. Stegun, editors, *Handbook of Mathematical Functions* (Dover Publications Inc., New York (1970)).

FIGURES

FIG. 1. $^1S_0$ partial wave.

FIG. 2. $^3P_0$ partial wave.

FIG. 3. $^3P_1$ partial wave.

FIG. 4. $^3P_2$ partial wave.

FIG. 5. $^3P_2 \longrightarrow {}^3F_2$ partial wave.

FIG. 6. $^3F_2 \longrightarrow {}^3P_2$ partial wave.

FIG. 7. $^3S_1$ partial wave.

FIG. 8. $^3S_1 \longrightarrow {}^3D_1$ partial wave.

FIG. 9. $^3D_1 \longrightarrow {}^3S_1$ partial wave.

FIG. 10. $^3D_1$ partial wave.

FIG. 11. $^1P_1$ partial wave.